Continuous Piecewise Polynomials and Static Equilibrium

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Mathematics Colloquium

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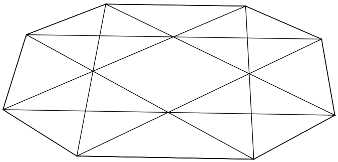

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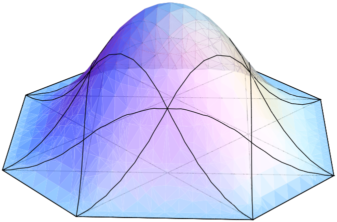

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Graph of the **Zwart-Powell element**: a spline in $C^1_2(\mathcal{P})$

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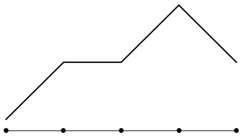

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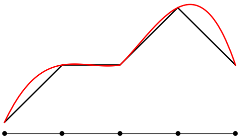

Trapezoid Rule!

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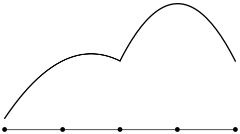

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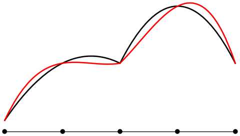

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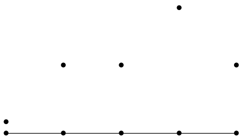

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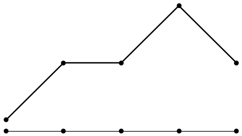

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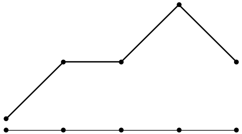

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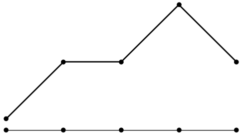

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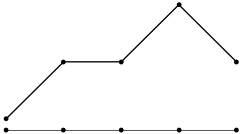

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Basis for $C^0_1(I)$:'Courant functions' or 'tent functions' which are 1 at a chosen vertex and 0 at all others.

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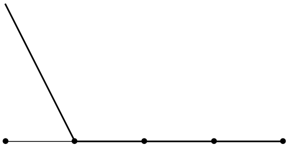
Univariate Courant functions:



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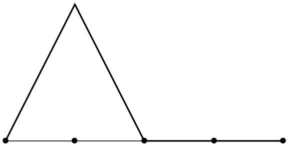

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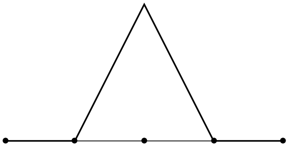

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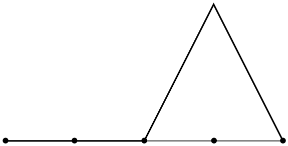

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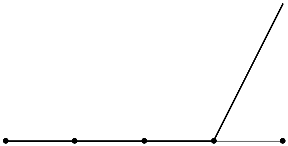

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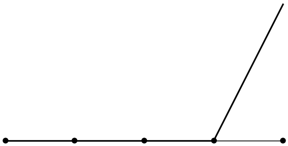

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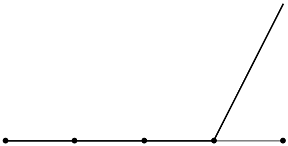

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Can generalize this dimension formula for all r, d:

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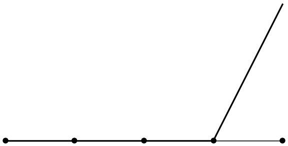

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There are nice algorithms due to Casteljau and de Boor to compute bases of $C_d^r(I)$ called **B-splines**.

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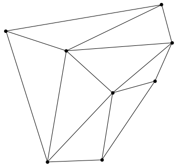

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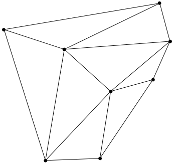

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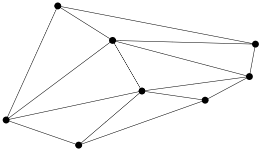
What kinds of PL functions are there on Δ?

Again, a continuous piecewise linear function on Δ is uniquely determined by its value on the vertices (3 points determine a plane!).



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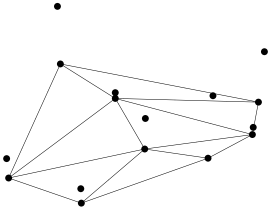

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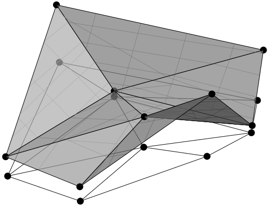

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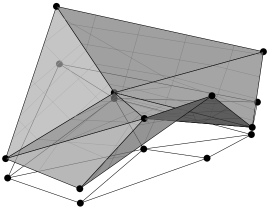

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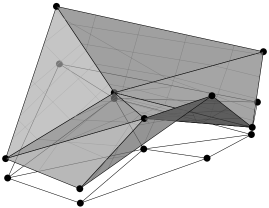

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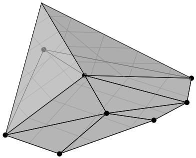

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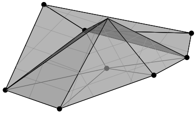

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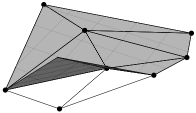

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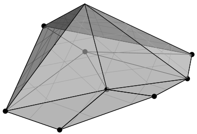

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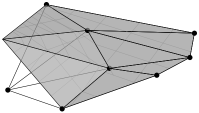

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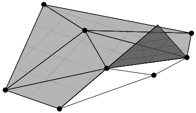

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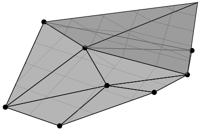

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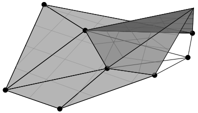

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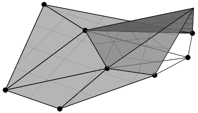

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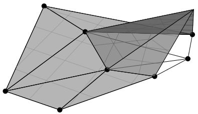

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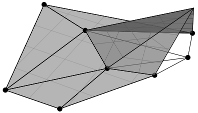

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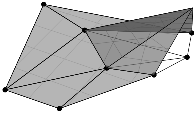

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There is no reference to the geometry of Δ! **All** that matters is the number of faces, edges, and vertices.

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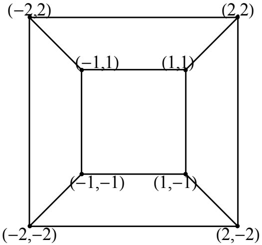

A polygonal framework $\mathcal{P}_1$ with
$f = 5, e = 12, v = 8, e^0 = 4, v^0 = 4$

I don't like triangles

Another way to make life more difficult (and interesting!) is to consider subdivisions of a simply connected domain by polygons.

- $\mathcal{P}$ denotes a subdivision of a disk by polygons
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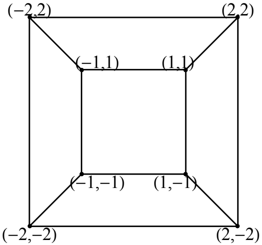

A polygonal framework $\mathcal{P}_1$ with
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What kinds of PL functions are there on $\mathcal{P}$?

Things get complicated

$\dim_{\mathbb{R}} C^0_1(\mathcal{P})$ **depends on the geometry** of $\mathcal{P}$!

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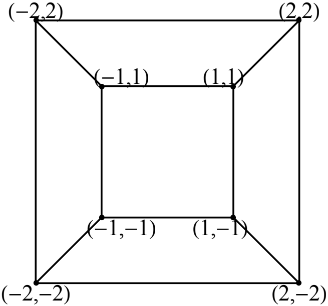

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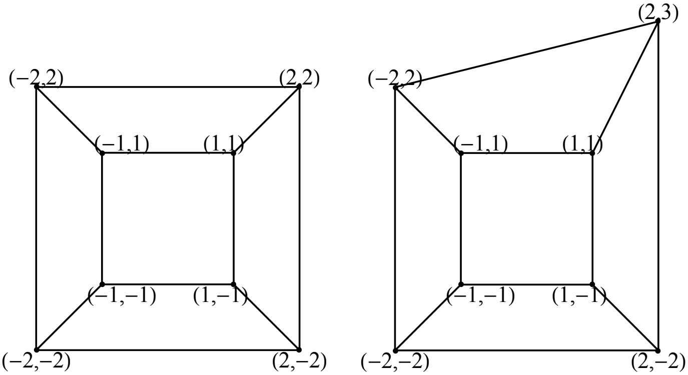

$\dim C^0_1(\mathcal{P}_1) = 4$ $\quad$ $\dim C^0_1(\mathcal{P}_2) = 3$

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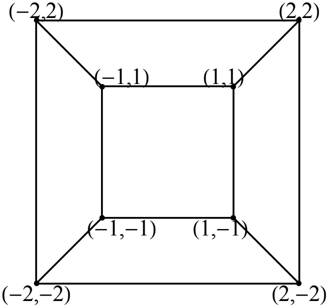

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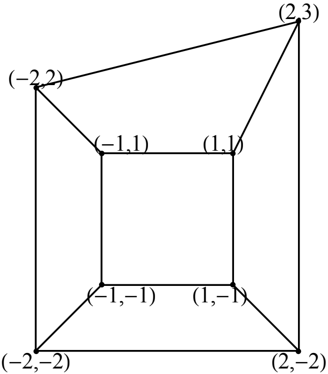

$\dim C^0_1(\mathcal{P}_2) = 3$

Let's see why.

- A **trivial** PL function is one which restricts to the same linear function on each face.

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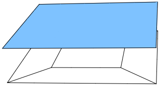

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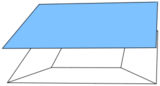

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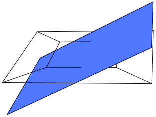

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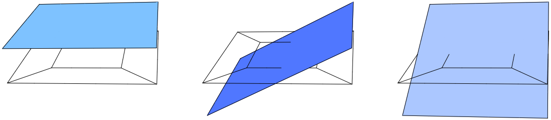

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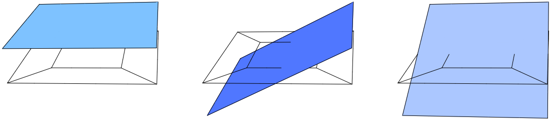

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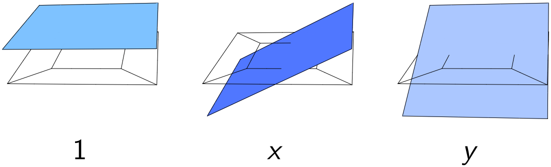

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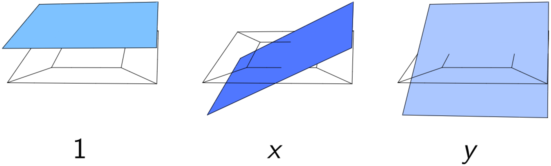

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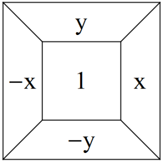

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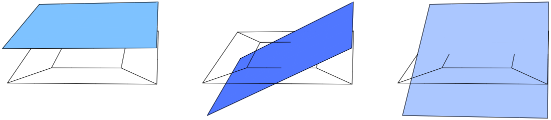

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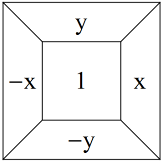

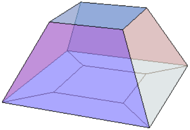

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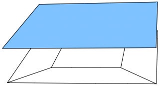

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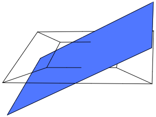

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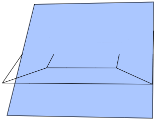

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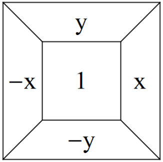

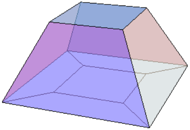

When you move to $\mathcal{P}_2$ you lose this PL function!

Generating Interesting Examples

Polygonal frameworks coming from a polytopes often have PL functions that are lost under small perturbations of the vertices.

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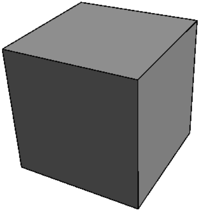
Here's a cube



Generating Interesting Examples

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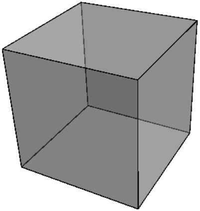
Make it transparent



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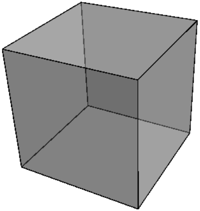
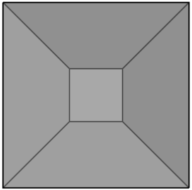
Polygonal frameworks coming from a polytopes often have PL functions that are lost under small perturbations of the vertices.

Make it transparent   Now look in one of the faces:





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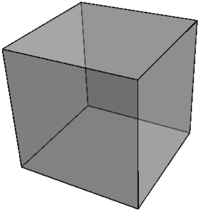
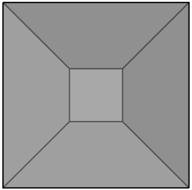
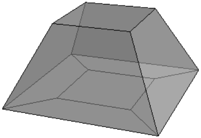


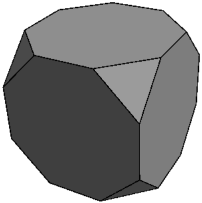




The nontrivial PL function is a ‘deformed cube’

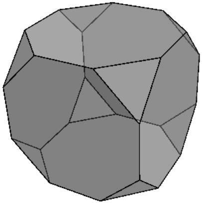
A more interesting example

Chop off cube corners



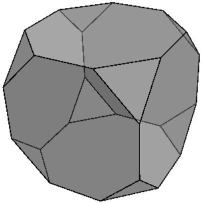
A more interesting example

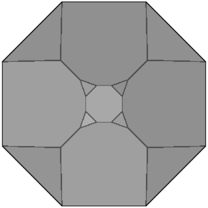
Make it transparent



A more interesting example

Make it transparent   Look into an octagonal face:





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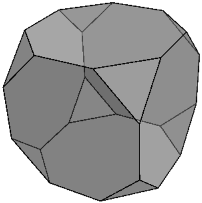

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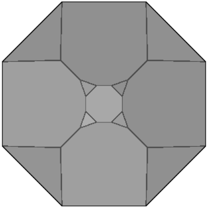

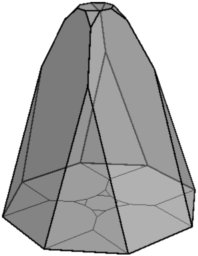

We get a nontrivial PL function which is a 'deformed' version of the truncated cube

And now for something completely different

- Framework of bars and joints represented by edges and vertices of polygonal framework

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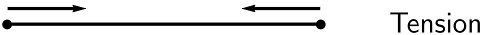

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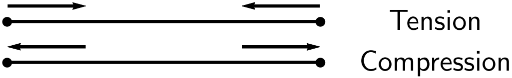

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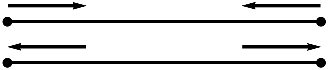

Tension

Compression

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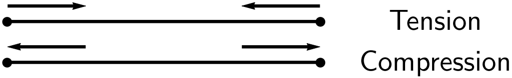

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Between vertices p_i, p_j (thought of as vectors), represent tension or compression as a scalar ω_{ij}.

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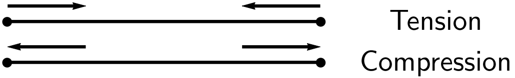

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- $\omega_{ij} < 0 \implies$ tension
- $\omega_{ij} > 0 \implies$ compression

Self-Stress

A **self-stress** on a framework is an assignment of scalars ω_{ij} along the edges e_{ij} satisfying

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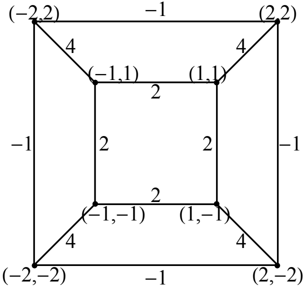

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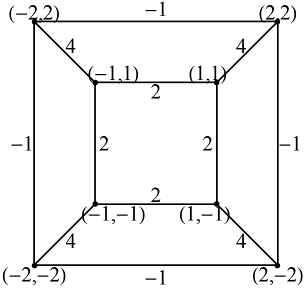

A nontrivial self-stress on $\mathcal{P}_1$

By the way, what could this mean physically?

Maxwell's Observation

Nontrivial stresses are in 1-1 correspondence (almost) with nontrivial PL functions on $\mathcal{P}$ which vanishes along the boundary!

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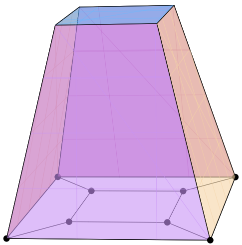

Start with graph

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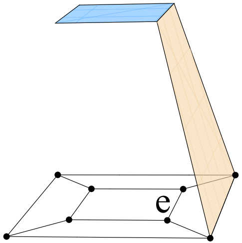

Restrict to faces adjacent to a single edge e

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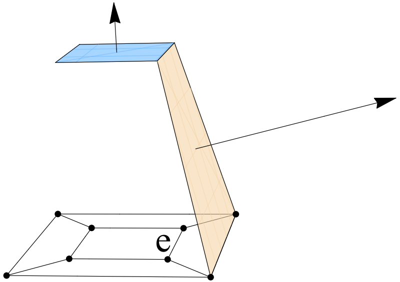

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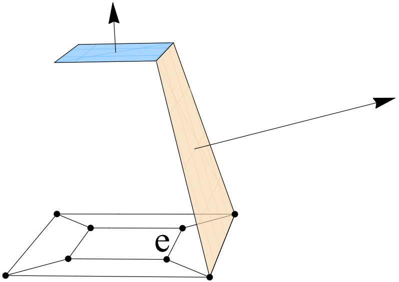

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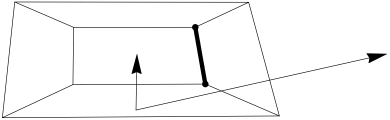

Translate normals to $(0, 0, -1)$

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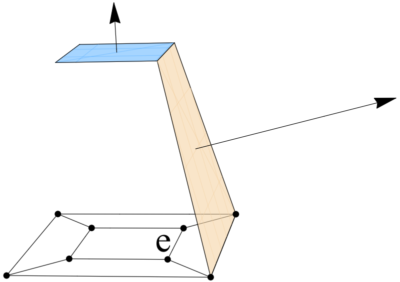

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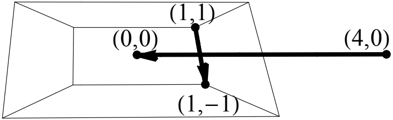
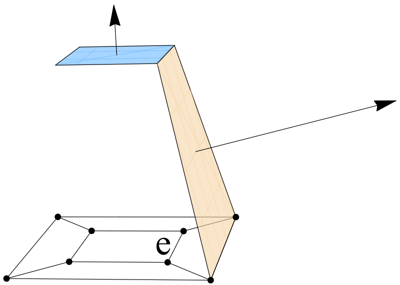


Translate normals to $(0, 0, -1)$

Connect normal tips

Maxwell's Observation

Nontrivial stresses are in 1-1 correspondence (almost) with nontrivial PL functions on $\mathcal{P}$ which vanishes along the boundary!



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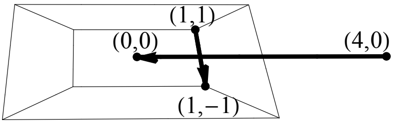

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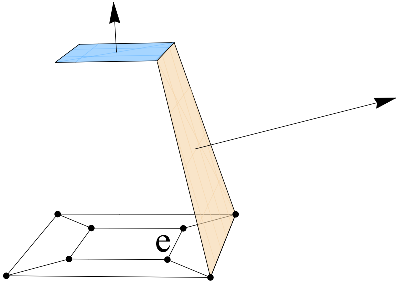

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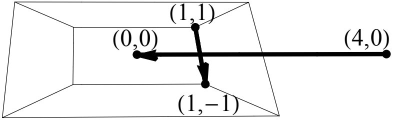

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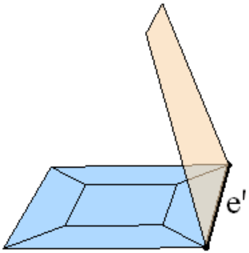
Sign of ω_e **depends on orientation**.

Flipping Orientations



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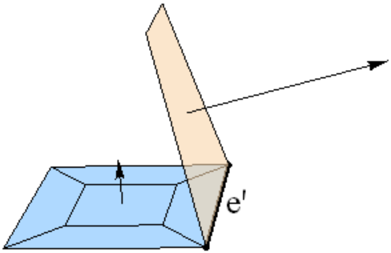

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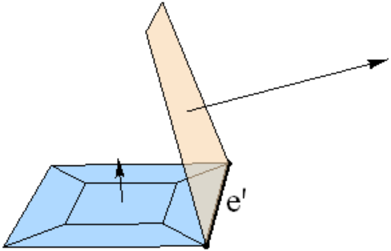

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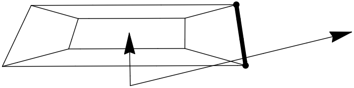

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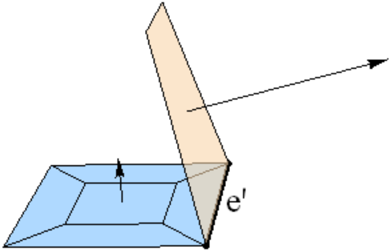

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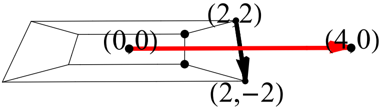

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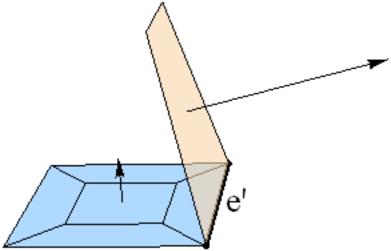

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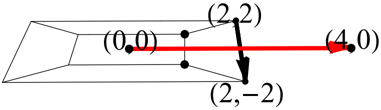

Translate normals to $(0,0,-1)$

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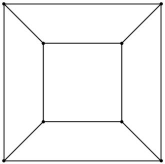
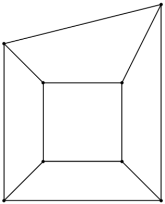


$\mathcal{P}_1$ **is not** independent



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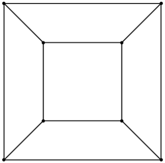

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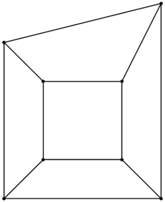

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Fact: If the domain is not simply connected, the above correspondence breaks down!

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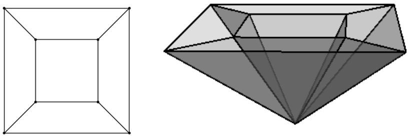

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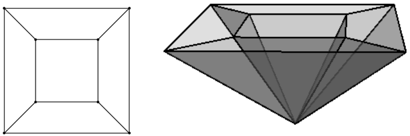

$\mathcal{P}_1$ $\quad$ $\widehat{\mathcal{P}}_1$

- $C^0(\widehat{\mathcal{P}})$ is **graded** (every spline can be written as a sum of splines of uniform degree)
- $C^0_d(\mathcal{P})$ 'sits inside' $C^r(\widehat{\mathcal{P}})$ as the degree d 'slice.'

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- $F \in C^0(\widehat{\mathcal{P}})$, $f \in \mathbb{R}[x, y, z]$ a polynomial. Then $f \cdot F \in C^0(\widehat{\mathcal{P}})$.
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- Via some homological algebra, $\dim C^0_1(\mathcal{P})$ has consequences for **freeness** of $C^0(\widehat{\mathcal{P}})$ as an $\mathbb{R}[x,y,z]$-module. This in turn impacts how easy it is to calculate $\dim C^0_d(\mathcal{P})$ for $d \geq 1$.

THANK YOU!