Commutative Algebra and Piecewise Polynomials

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Colloquium

Piecewise Polynomials

Spline

A piecewise polynomial function, continuously differentiable to some order.

Some Context: Splines in Calculus 1

Low degree splines are used in Calc 1 to approximate integrals.

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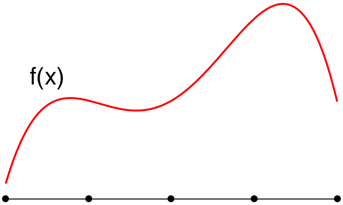

Graph of a function

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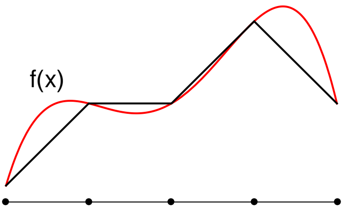

Trapezoid Rule

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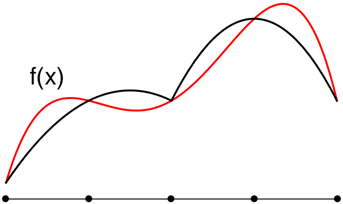
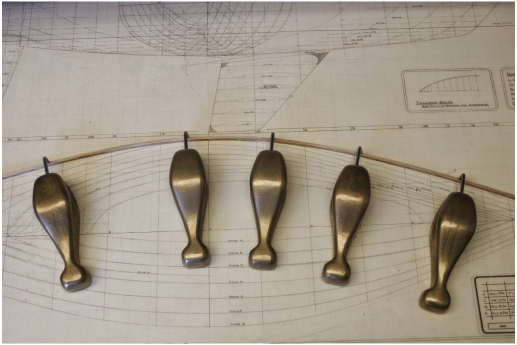


Simpson's Rule

Origin

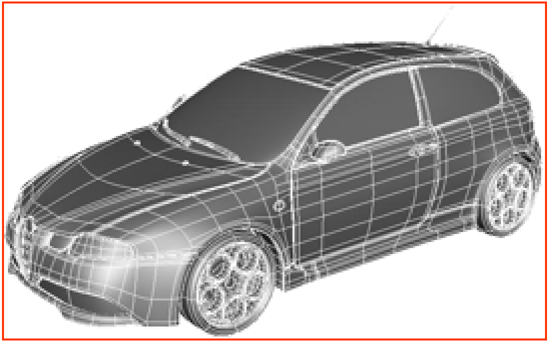
Term **spline** originated in shipbuilding - referred to flexible wooden strips anchored at several points.



Source: *http://technologycultureboats.blogspot.com/2014/12/gustave-caillebotte-and-curves.html*

Current applications

- Computer-Aided Geometric Design (CAGD): splines used to create models by interpolating datapoints.



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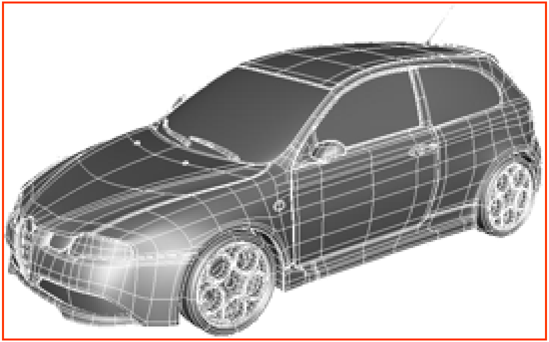

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- Finite Element Method (FEM): best approximation to a solution of a partial differential equation (PDE) is obtained in a spline space
- FEM especially useful for PDEs in engineering and mathematical physics

Univariate splines

Most widely studied case: approximation of a function $f(x)$ over an interval $\Delta = [a, b] \subset \mathbb{R}$ by C^r piecewise polynomials.

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- Subdivide $\Delta = [a, b]$ into subintervals:
  $\Delta = [a_0, a_1] \cup [a_1, a_2] \cup \cdots \cup [a_{n-1}, a_n]$
- Find a basis for the vector space $C^r_d(\Delta)$ of C^r piecewise polynomial functions on Δ with degree at most d (e.g. B-splines)
- Find best approximation to $f(x)$ in $C^r_d(\Delta)$

Example: two subintervals

$\Delta = [a_0, a_1] \cup [a_1, a_2]$ (assume WLOG $a_1 = 0$)

$$\begin{aligned} (f_1, f_2) \in C^r_d(\Delta) &\iff f_1^{(i)}(0) = f_2^{(i)}(0) \text{ for } 0 \leq i \leq r \\ &\iff x^{r+1} | (f_2 - f_1) \\ &\iff (f_2 - f_1) \in \langle x^{r+1} \rangle \end{aligned}$$

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Even more explicitly:

- $f_1(x) = b_0 + b_1 x + \cdots + b_d x^d$
- $f_2(x) = c_0 + c_1 x + \cdots + c_d x^d$
- $(f_1, f_2) \in C^r_d(\Delta) \iff b_0 = c_0, \ldots, b_r = c_r.$

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$$\dim C^r_d(\Delta) = \begin{cases} d+1 & \text{if } d \leq r \\ (d+1) + (d-r) & \text{if } d > r \end{cases}$$

Note: $\dim C^r_d(\Delta)$ is polynomial in d for $d > r$.

Dimension two

Subdivision $I \subset \mathbb{R}^1 \rightarrow$ **polytopal complex** $\Delta \subset \mathbb{R}^2$

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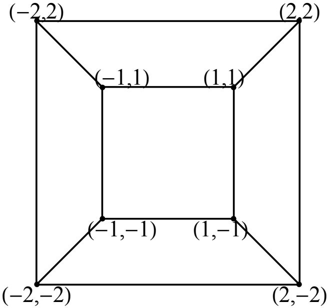

A polytopal complex $\mathcal{Q}$

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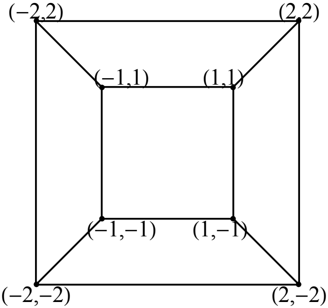

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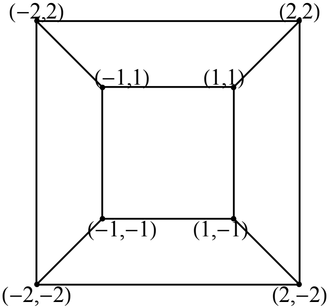

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- Δ : full dimensional convex polygons (polytopes) 'glued' together along faces to yield a domain Ω with no holes
- $C^r(\Delta)$: piecewise polynomial functions on Δ which are continuously differentiable of order r (C^r splines)
- $C^r_d(\Delta)$: C^r splines of degree at most d

The dimension question

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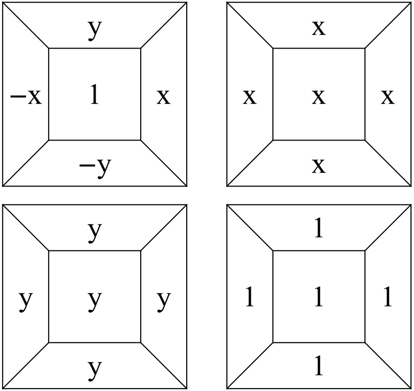

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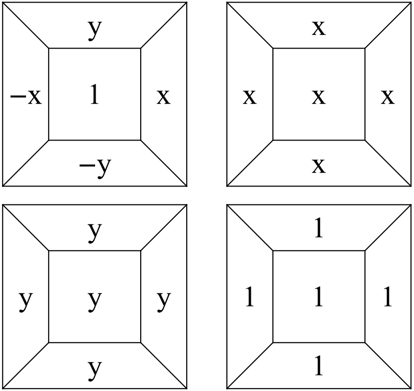

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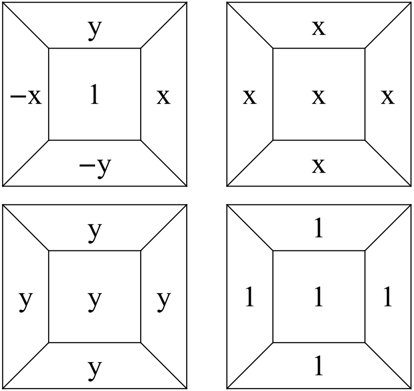

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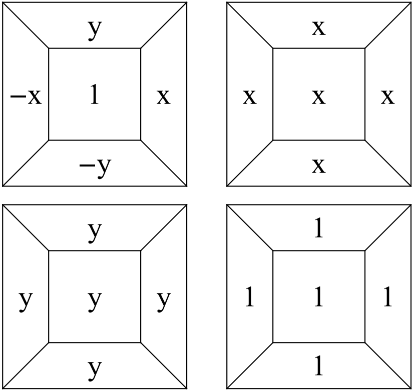

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Posed in 1973 by Strang for C^1 splines on triangulations

Translation to algebra

(Algebraic) Spline Criterion:

- If τ is an edge of Δ, $l_\tau =$ affine form vanishing on affine span of τ
- Collection $\{F_\sigma\}$ (one for each 2-dimensional polytope σ) define $F \in C^r(\Delta) \iff$ for every pair of adjacent polytopes $\sigma_1, \sigma_2 \in \Delta_2$ with $\sigma_1 \cap \sigma_2 = \tau$, $l_\tau^{r+1} \mid (F_{\sigma_1} - F_{\sigma_2})$

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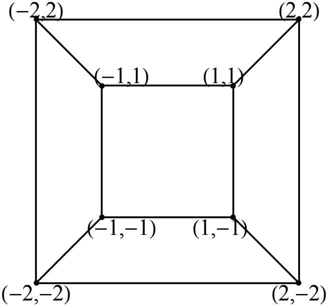

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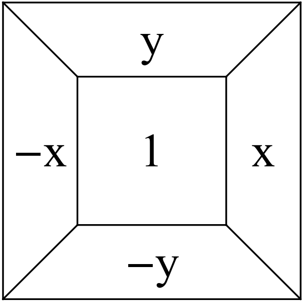

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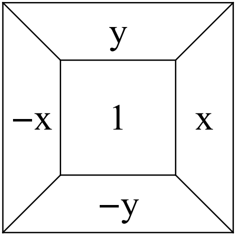

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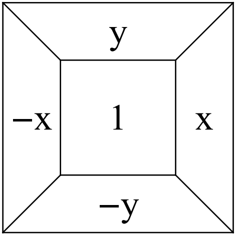

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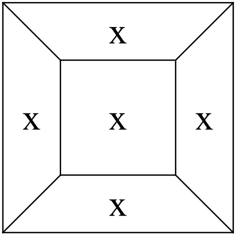

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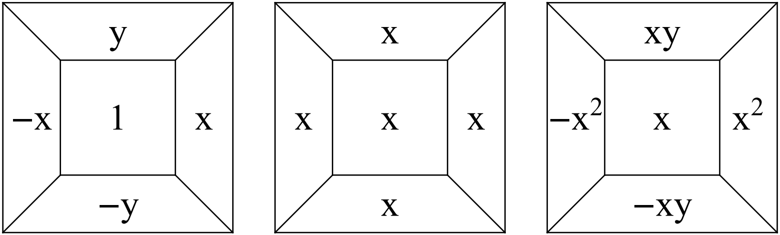
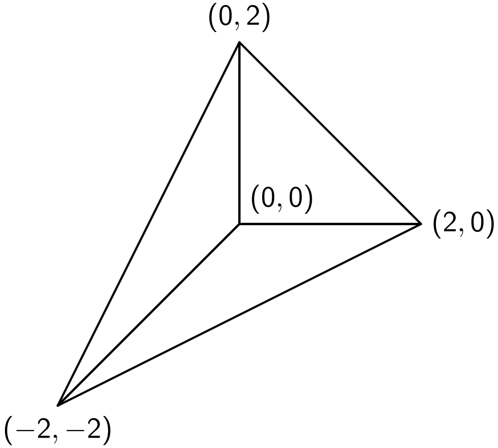


$F \in C^0_1(\mathcal{Q})$ $\quad x \in C^0_1(\mathcal{Q})$ $\quad xF \in C^0_2(\mathcal{Q})$

Example: continuous splines on a simplicial complex



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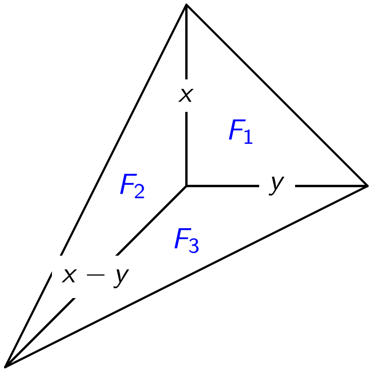

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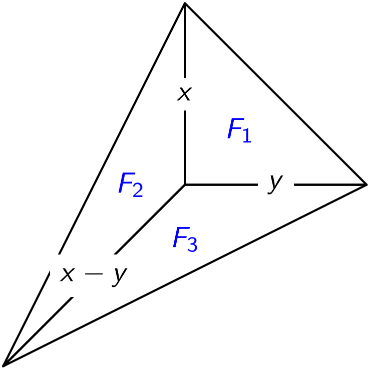
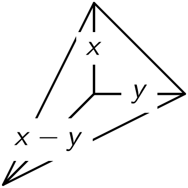


$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \in C^0(\Delta) \iff \exists f_1, f_2, f_3$$

so that

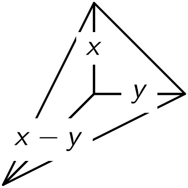
$$\begin{aligned} F_1 - F_2 &= f_1 x \\ F_2 - F_3 &= f_2(x - y) \\ F_3 - F_1 &= f_3 y \end{aligned}$$

Example, continued: freeness



Three splines in $C^0(\Delta)$:

Example, continued: freeness



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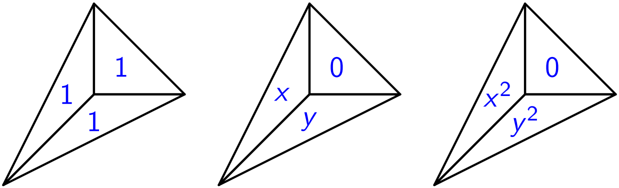

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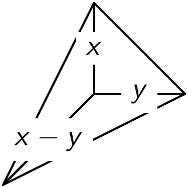

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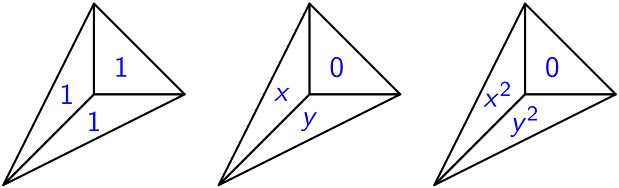

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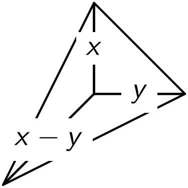

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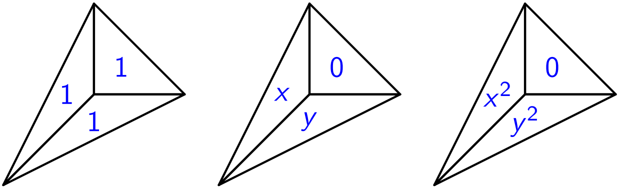

- In fact, every spline $F \in C^0(\Delta)$ can be written uniquely as a polynomial combination of these three splines.
- We say $C^0(\Delta)$ is a **free** $\mathbb{R}[x, y]$-module

Example, continued: dimension computation

$$C_d^0(\Delta) \cong \mathbb{R}[x,y]_{\le d} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \oplus \mathbb{R}[x,y]_{\le d-1} \begin{pmatrix} 0 \\ x \\ y \end{pmatrix} \oplus \mathbb{R}[x,y]_{\le d-2} \begin{pmatrix} 0 \\ x^2 \\ y^2 \end{pmatrix}$$

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$$\begin{aligned} \mathbb{R}[x,y]_{\leq k} &= \mathrm{span}\{x^i y^j : i+j \leq k\} \\ \dim \mathbb{R}[x,y]_{\leq k} &= \tbinom{k+2}{2} \end{aligned}$$

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$$\dim C^0_d(\Delta) = \binom{d+2}{2} + \binom{d+1}{2} + \binom{d}{2}$$

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\dim C^0_d(\Delta) = &\quad \binom{d+2}{2} + \binom{d+1}{2} + \binom{d}{2} \\
= &\quad \tfrac{3}{2}d^2 + \tfrac{3}{2}d + 1 \text{ for } d \ge 0
\end{aligned}$$

Hilbert series and polynomial

$\Delta \subset \mathbb{R}^2$. From commutative algebra

- $\dim C^r_d(\Delta)$ is called the **Hilbert function** of $C^r(\Delta)$
- Hilbert function is eventually a polynomial of degree 2 in d, called the **Hilbert polynomial** of $C^r(\Delta)$ and denoted $HP(C^r(\Delta), d)$

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$$HS(C^r(\Delta), t) = \frac{h(t)}{(1-t)^3}, \text{ where } h(t) \in \mathbb{Z}[t].$$

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- What is a formula for $HP(C^r(\Delta), d)$?

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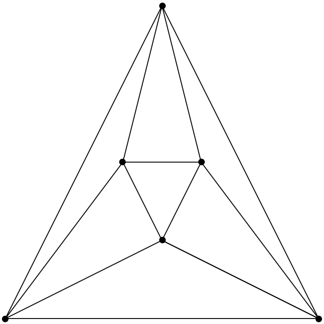
- Determine $HS(C^r(\Delta), t)$. (too hard!)
- What is a formula for $HP(C^r(\Delta), d)$?
- How large must d be so that $\dim C^r_d(\Delta) = HP(C^r(\Delta), d)$?

Low Degree: Morgan-Scot triangulation



$\dim C^1_2(\mathcal{T}) = 7$

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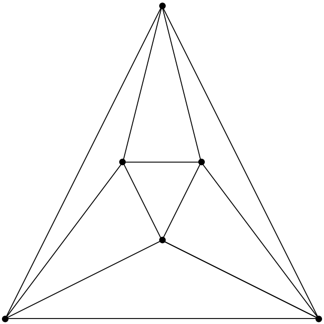

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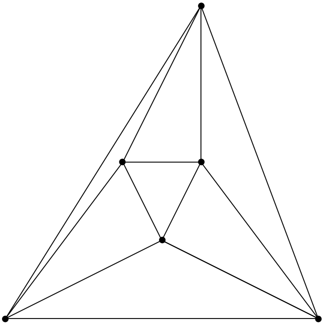

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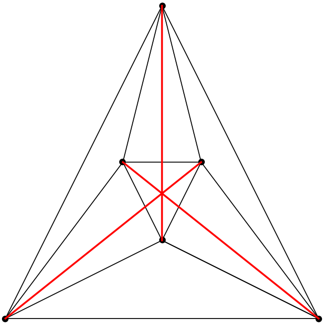

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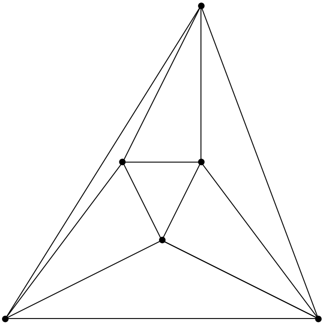

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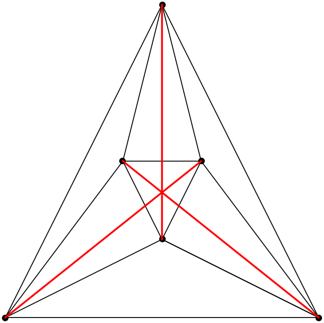

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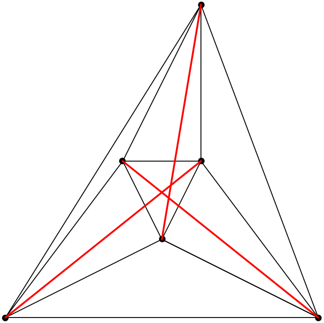

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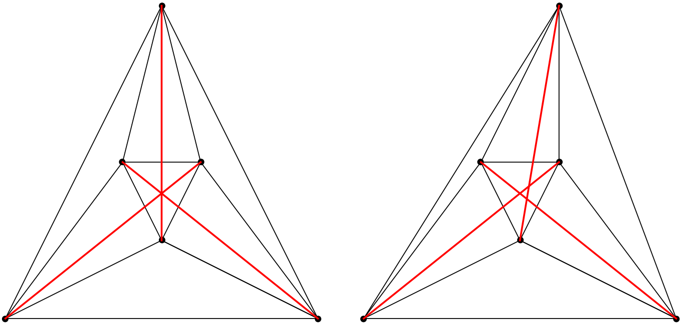

$\dim C_2^1(\mathcal{T}) = 7$ $\dim C_2^1(\mathcal{T}') = 6$

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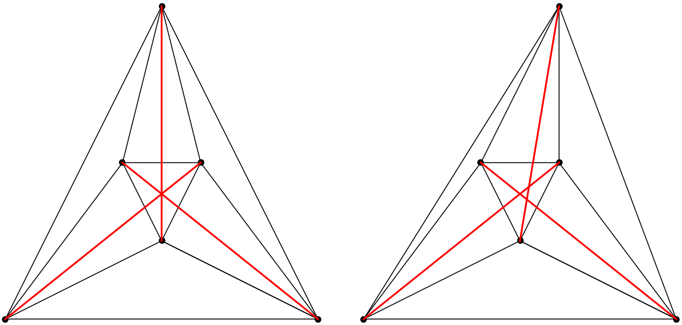

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$\dim C_d^1(\mathcal{T}) = \dim C_d^1(\mathcal{T}')$ if $d \neq 2$!

Conjecture (at least 30 years old)

$\dim C_d^1(\Delta) = HP(C^1(\Delta), d)$ for $d \geq 3$.

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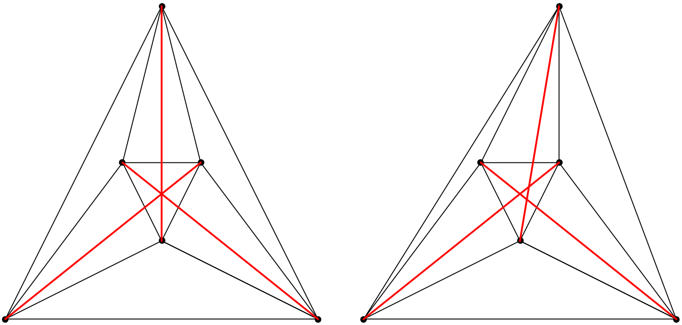

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Only $\dim C^1_2(\Delta)$ can differ from expected dimension formula

Piecewise linear functions

If $\Delta \subset \mathbb{R}^2$ is a triangulation with v vertices, then $\dim C_1^0(\Delta) = v$.

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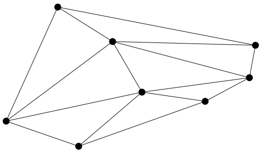

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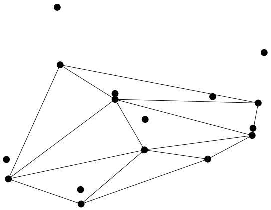

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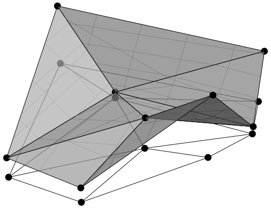

Tent functions

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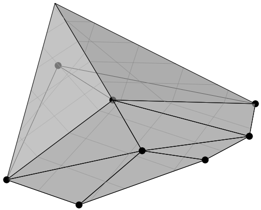

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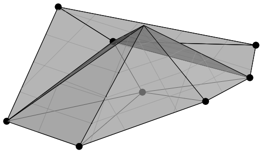

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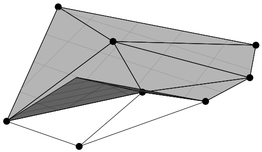

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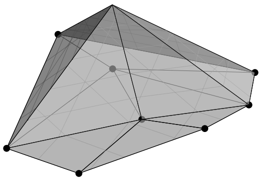

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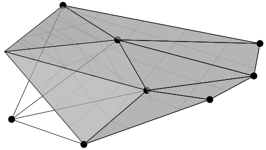

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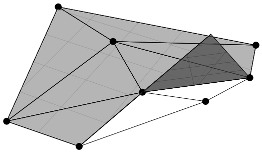

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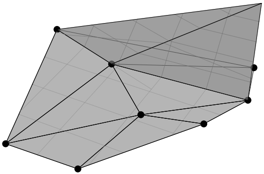

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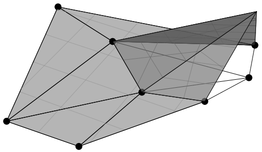

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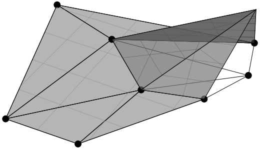

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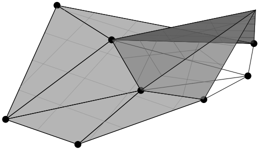

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- [Billera '89]: If $\Delta \subset \mathbb{R}^2$ is a triangulation of a domain Ω without holes then
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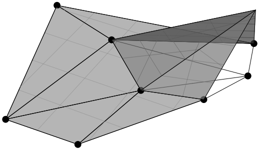

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Generalizes to arbitrary dimensions.

PL functions depend on geometry

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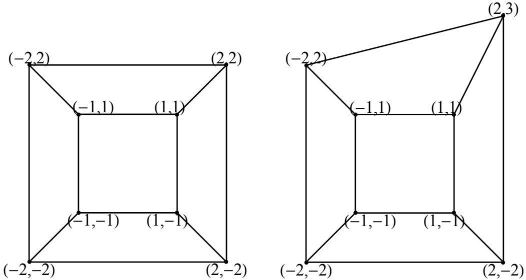
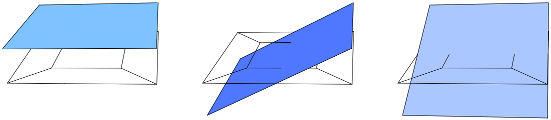


$\mathcal{Q}_1$ $\qquad$ $\mathcal{Q}_2$

$\dim C_1^0(\mathcal{Q}_1) = 4$ $\qquad$ $\dim C_1^0(\mathcal{Q}_2) = 3$

Trivial PL Functions

- A **trivial** PL function on $\Delta \subset \mathbb{R}^2$ has the same linear function on each face.
- dim(trivial splines on Δ) = 3 **always**, with basis $1, x, y$.



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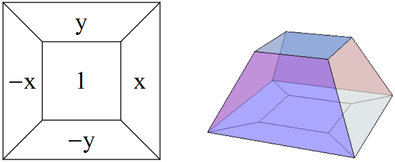

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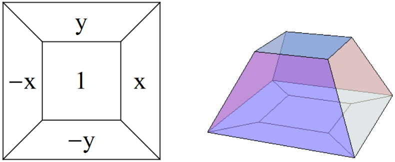

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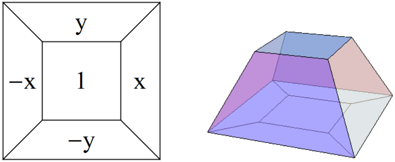

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- In general, computing $C^0_1(\Delta)$ entails determining when edges of Δ come from projecting a polyhedral surface
- Relates to *rigidity theory*, dates back to Maxwell in 1860s

Impact of PL functions on freeness

Nonfreeness for Polytopal Complexes [D. ‘12]

If Δ is a polytopal subdivision of a planar domain Ω without holes, $C^0(\Delta)$ need not be free [D. ‘12].

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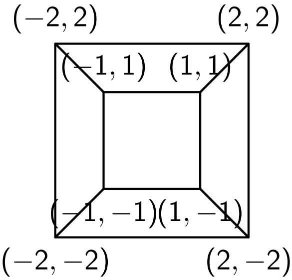

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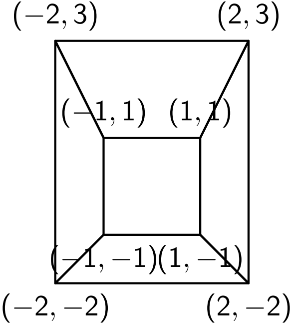

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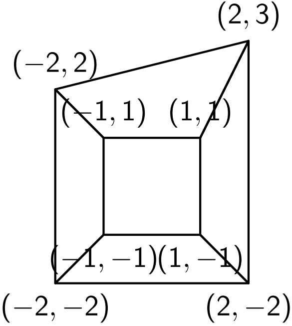

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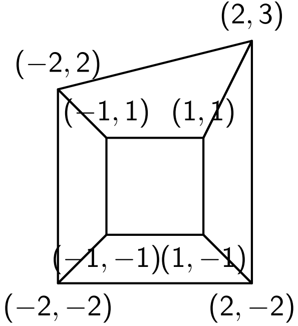

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Persistence of 'extra' PL function determines freeness.

Planar simplicial splines of large degree

Planar simplicial dimension [Alfeld-Schumaker '90]

If $\Delta \subset \mathbb{R}^2$ is a simply connected triangulation and $d \geq 3r+1$,

$$\dim C^r_d(\Delta) = f_2\binom{d+2}{2} - f_1^0\left(\binom{d+2}{2} - \binom{d-r+1}{2}\right) + \sigma,$$

- $f_i(f_i^0)$ is the number of i-faces (interior i-faces).
- $\sigma =$ constant obtained as a sum of contributions from each interior vertex.

Planar non-simplicial splines of large degree

Planar non-simplicial dimension [McDonald-Schenck '09]

If $\Delta \subset \mathbb{R}^2$ is a simply connected polytopal complex and $d \gg 0$,

$$\dim C^r_d(\Delta) = f_2 \binom{d+2}{2} - f_1^0 \left(\binom{d+2}{2} - \binom{d-r+1}{2} \right) + \sigma + \sigma',$$

- $f_i (f_i^0)$ is the number of i-faces (interior i-faces).
- σ = sum of constant contributions from interior vertices
- σ' = sum of constant contributions from 'missing' vertices

Agreement for non-simplicial splines

How large must d be in order for $HP(C^r(\Delta), d) = \dim C^r_d(\Delta)$?

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- Worst case examples indicate bound in theorem is off by a factor of about two (F is necessary!)
- Best known bounds in simplicial case are also off by a factor of 1.5
- Proof of theorem uses notion of regularity from algebraic geometry

Curved Partitions

More general problem: Compute $\dim C^r_d(\Delta)$ where Δ is a planar partition whose arcs consist of irreducible algebraic curves.

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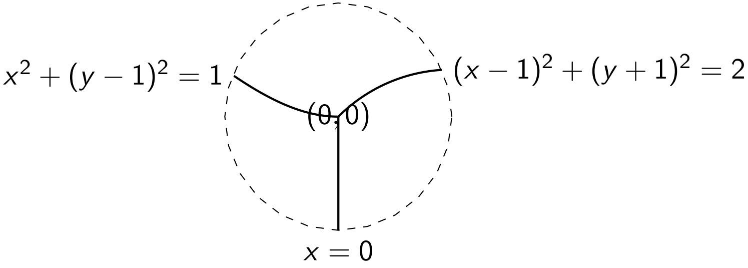

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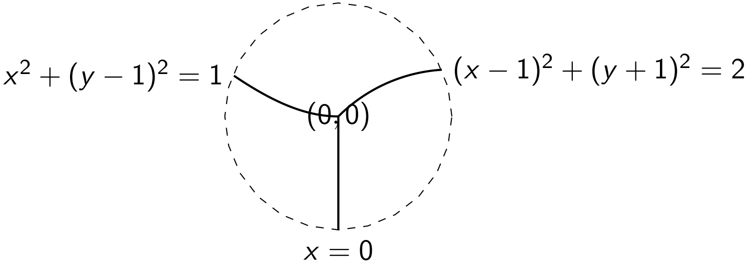
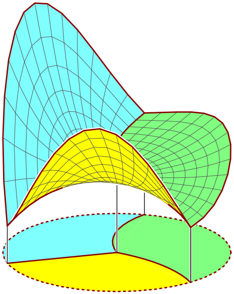


Call functions in $C^r(\Delta)$ *semi-algebraic splines* since they are defined over regions given by polynomial inequalities, or semi-algebraic sets.

Graphs of some semi-algebraic splines



Graph of a spline in $C_3^0(\Delta)$

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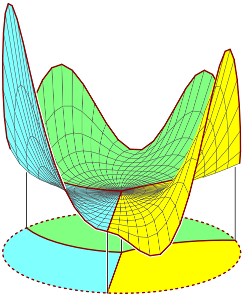

Graph of a spline in $C^1_6(\Delta)$

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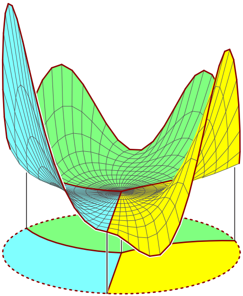

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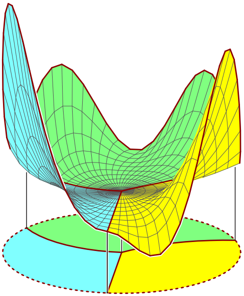

Graph of a spline in $C^1_6(\Delta)$

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- Recent work suggests semi-algebraic splines may be increasingly useful in finite element method [Davydov-Kostin-Saeed '16]

Linearizing

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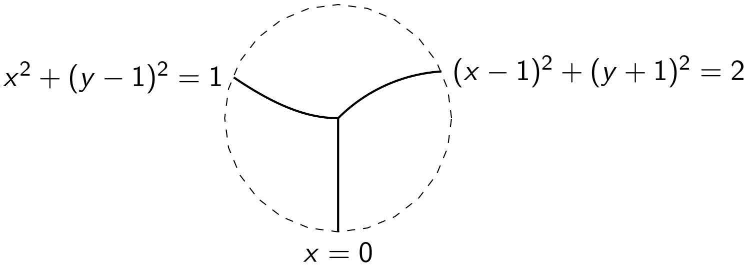

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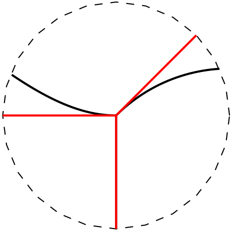

Tangent rays

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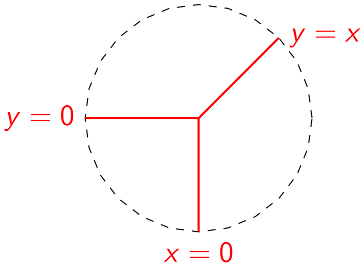

Δ_L

Linearizing the local case

Theorem: Linearizing $\dim C^r_d(\Delta)$ [D.-Sottile-Sun '17]

Let Δ consist of n irreducible curves of degree $d_1, \ldots, d_n$ meeting at $(0,0)$ with distinct tangents and no common zero in $\mathbb{P}^2(\mathbb{C})$ other than $(0,0)$. Then, for $d \gg 0$,

$$\dim C^r_d(\Delta) = \dim C^r_d(\Delta_L) + \sum_{i=1}^{n} \left(\binom{d+2-d_i(r+1)}{2} - \binom{d-r-1}{2} \right)$$

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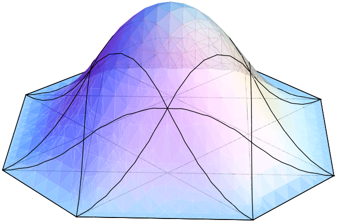
- Bounds on d for when equality holds are also considered, using regularity

THANK YOU!



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- If $\Delta \subset \mathbb{R}^3$, $\dim C^r_d(\Delta)$ is not known for $d \gg 0$ except for $r=1$, $d \geq 8$ on generic triangulations [Alfeld-Schumaker-Whitely '93]. (connects to unsolved problems in algebraic geometry)

Open Questions

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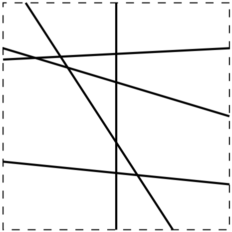

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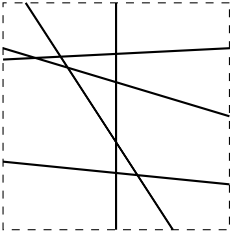

- Basis for $C^r_d(\Delta)$ and dim $C^r_d(\Delta)$ [Chui-Wang '83]
- $C^r(\Delta)$ is free for any r [Schenck '97]

Ziegler's Pair

$\mathcal{A}_t$ = union of planes defined by vanishing of the nine linear forms:

x	y	x+y+z	2x+y+z	(1+t)x+(3+t)z
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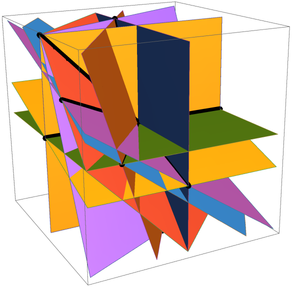

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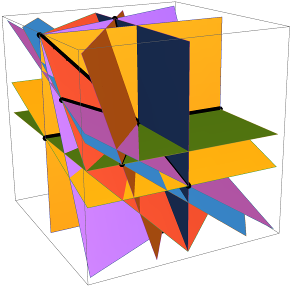

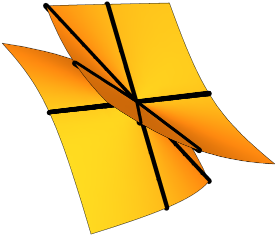

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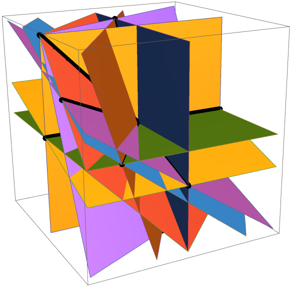

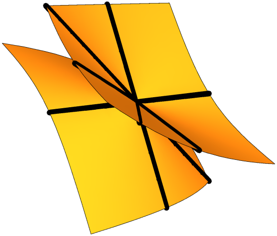

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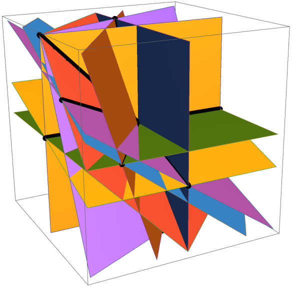

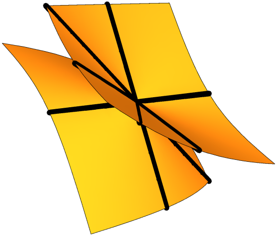

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- $C^0(\Delta_t)$ is free if and only if $t \neq 0$!