Piecewise Linear Functions, Projecting Polytopes, and Equilibrium Stresses

Michael DiPasquale
Colorado State University

Universidad Michoacana de San Nicolás de Hidalgo

Piecewise Linear Functions (PL Functions)

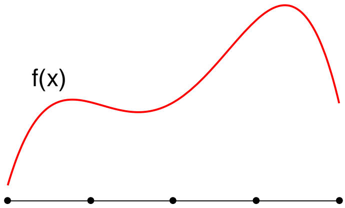
A function which is continuous and piecewise linear over some subdivision.

PL functions in Calculus

Piecewise linear (PL) functions are used in calculus to approximate integrals.

PL functions in Calculus

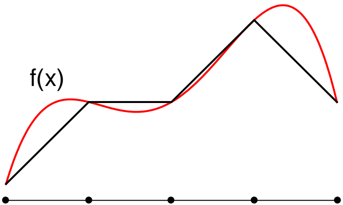
Piecewise linear (PL) functions are used in calculus to approximate integrals.



Graph of a function

PL functions in Calculus

Piecewise linear (PL) functions are used in calculus to approximate integrals.



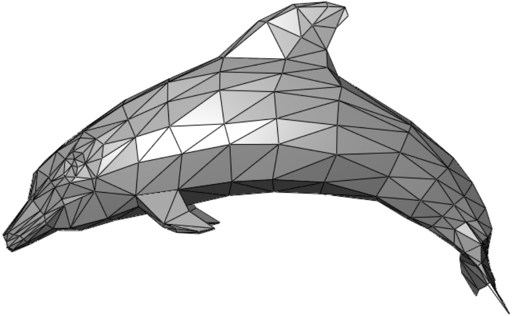
Trapezoid Rule

Applications: Computer-Aided Geometric Design

Piecewise linear (PL) functions are used to create models of complex objects.

Applications: Computer-Aided Geometric Design

Piecewise linear (PL) functions are used to create models of complex objects.



Source: *http://en.wikipedia.org/wiki/File:Dolphin_triangle_mesh.png*

Counting PL Functions in one variable

$\Delta = [-1, 0] \cup [0, 1]$

$$h(x) = \begin{cases} ax + b & -1 \leq x < 0 \\ cx + d & 0 \leq x \leq 1 \end{cases}$$

Counting PL Functions in one variable

$\Delta = [-1, 0] \cup [0, 1]$

$$h(x) = \begin{cases} ax + b & -1 \leq x < 0 \\ cx + d & 0 \leq x \leq 1 \end{cases}$$

Which of the coefficients a, b, c, d can be chosen freely if $h(x)$ is required to be continuous?

Counting PL Functions in one variable

$\Delta = [-1, 0] \cup [0, 1]$

$$h(x) = \begin{cases} ax + b & -1 \leq x < 0 \\ cx + d & 0 \leq x \leq 1 \end{cases}$$

Which of the coefficients a, b, c, d can be chosen freely if $h(x)$ is required to be continuous?

- Plugging in $x = 0$ gives $b = d$

Counting PL Functions in one variable

$\Delta = [-1, 0] \cup [0, 1]$

$$h(x) = \begin{cases} ax + b & -1 \leq x < 0 \\ cx + d & 0 \leq x \leq 1 \end{cases}$$

Which of the coefficients a, b, c, d can be chosen freely if $h(x)$ is required to be continuous?

- Plugging in $x = 0$ gives $b = d$
- So free to determine a, b, c

Counting PL Functions in one variable

$\Delta = [-1, 0] \cup [0, 1]$

$$h(x) = \begin{cases} ax + b & -1 \leq x < 0 \\ cx + d & 0 \leq x \leq 1 \end{cases}$$

Which of the coefficients a, b, c, d can be chosen freely if $h(x)$ is required to be continuous?

- Plugging in $x = 0$ gives $b = d$
- So free to determine a, b, c
- PL functions on Δ are a **three dimensional** vector space

Dimension Question

If Δ is a union of subintervals,

- What is the dimension of the vector space of PL functions on Δ?

Dimension Question

If Δ is a union of subintervals,

- What is the dimension of the vector space of PL functions on Δ?
- In other words, how many free variables are there?

Dimension Question

If Δ is a union of subintervals,

- What is the dimension of the vector space of PL functions on Δ?
- In other words, how many free variables are there?
- Can we find a basis for this vector space?

Dimension Question

If Δ is a union of subintervals,

- What is the dimension of the vector space of PL functions on Δ?
- In other words, how many free variables are there?
- Can we find a basis for this vector space?

This question has its origins in approximation theory.

Counting Univariate PL Functions

The dimension of the space of PL functions on a subdivision is equal to the number of vertices of the subdivision.

Counting Univariate PL Functions

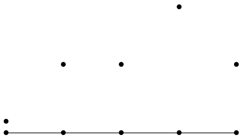
The dimension of the space of PL functions on a subdivision is equal to the number of vertices of the subdivision.

Proof by picture: PL function determined uniquely by value on **vertices**

Counting Univariate PL Functions

The dimension of the space of PL functions on a subdivision is equal to the number of vertices of the subdivision.

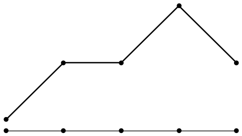
Proof by picture: PL function determined uniquely by value on **vertices**



Counting Univariate PL Functions

The dimension of the space of PL functions on a subdivision is equal to the number of vertices of the subdivision.

Proof by picture: PL function determined uniquely by value on **vertices**

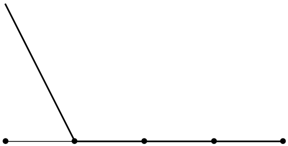


Tent Functions

A basis for PL functions is given by ‘Courant functions’ or ‘tent functions’ are 1 at a chosen vertex and 0 at all others.

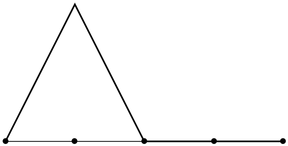
Tent Functions

A basis for PL functions is given by 'Courant functions' or 'tent functions' are 1 at a chosen vertex and 0 at all others.



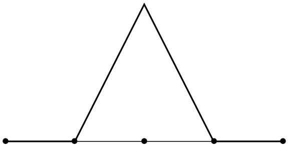
Tent Functions

A basis for PL functions is given by ‘Courant functions’ or ‘tent functions’ are 1 at a chosen vertex and 0 at all others.



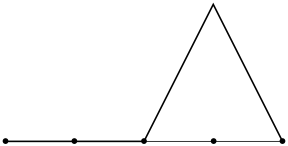
Tent Functions

A basis for PL functions is given by 'Courant functions' or 'tent functions' are 1 at a chosen vertex and 0 at all others.



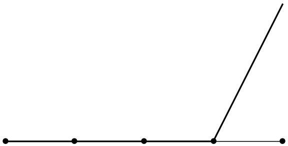
Tent Functions

A basis for PL functions is given by ‘Courant functions’ or ‘tent functions’ are 1 at a chosen vertex and 0 at all others.



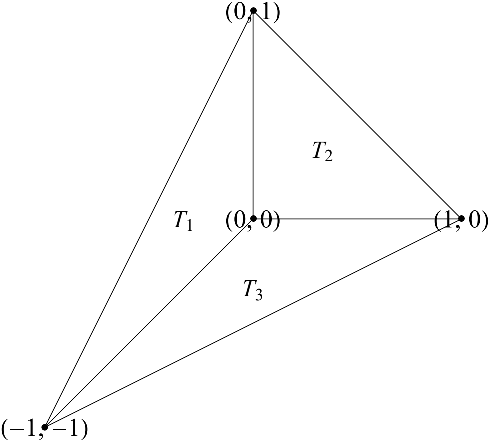
Tent Functions

A basis for PL functions is given by ‘Courant functions’ or ‘tent functions’ are 1 at a chosen vertex and 0 at all others.



Counting Bivariate PL Functions

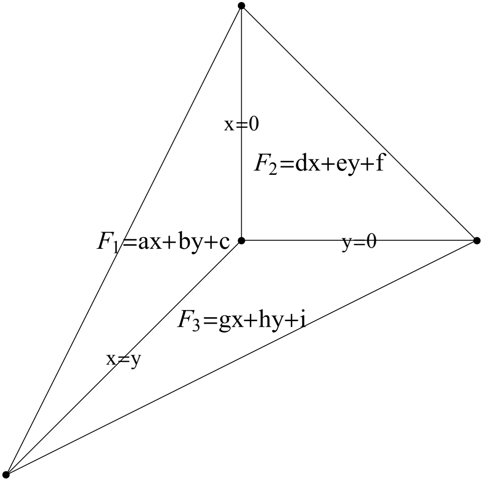
Δ = union of three triangles below



Δ

Counting Bivariate PL Functions

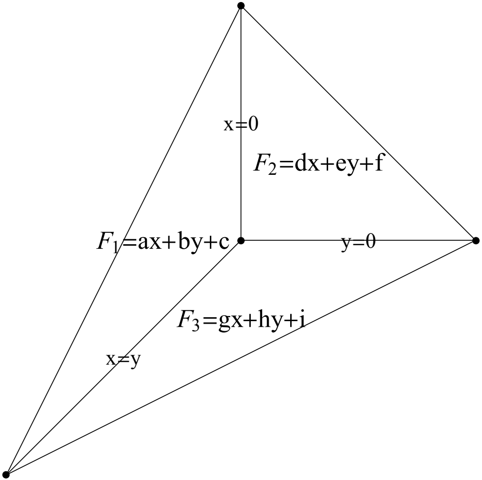
Δ = union of three triangles below



Candidate for PL function

Counting Bivariate PL Functions

Δ = union of three triangles below



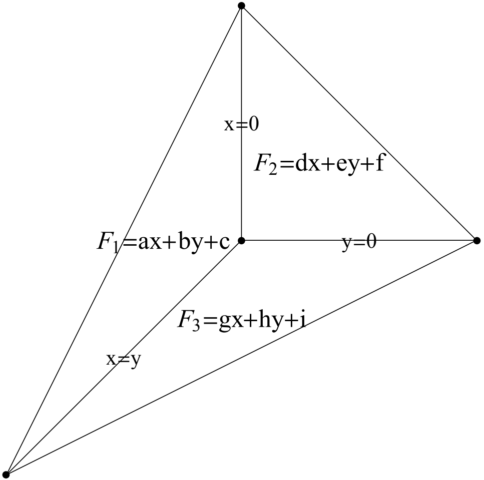
Candidate for PL function

Continuity $\Longrightarrow$

$$b = e$$
$$c = f = i$$
$$d = g$$
$$a + b = g + h$$

Counting Bivariate PL Functions

Δ = union of three triangles below



Candidate for PL function

Continuity $\Longrightarrow$

$$b = e$$
$$c = f = i$$
$$d = g$$
$$a + b = g + h$$

a, b, c, d determine e, f, g, h, i $\Longrightarrow$ PL functions is 4-dim vector space

Counting Bivariate PL Functions

If Δ is a planar triangulation with v vertices, then the space of PL functions has dimension equal to the number of vertices.

Counting Bivariate PL Functions

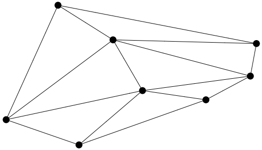
If Δ is a planar triangulation with v vertices, then the space of PL functions has dimension equal to the number of vertices.

Proof by picture: PL function on Δ uniquely determined by value at vertices.

Counting Bivariate PL Functions

If Δ is a planar triangulation with v vertices, then the space of PL functions has dimension equal to the number of vertices.

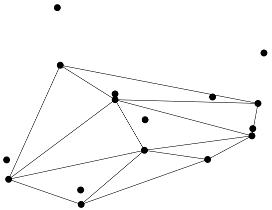
Proof by picture: PL function on Δ uniquely determined by value at vertices.



Counting Bivariate PL Functions

If Δ is a planar triangulation with v vertices, then the space of PL functions has dimension equal to the number of vertices.

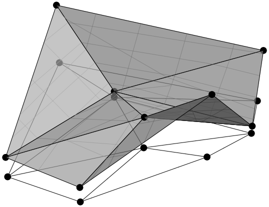
Proof by picture: PL function on Δ uniquely determined by value at vertices.



Counting Bivariate PL Functions

If Δ is a planar triangulation with v vertices, then the space of PL functions has dimension equal to the number of vertices.

Proof by picture: PL function on Δ uniquely determined by value at vertices.

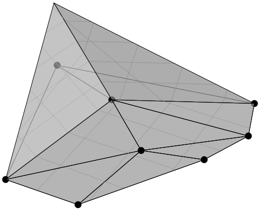


Tent Functions 2

A basis for PL functions is given by Courant functions, which take a value of 1 at a chosen vertex and 0 at all other vertices.

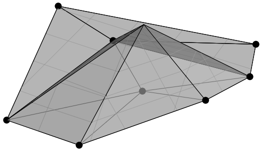
Tent Functions 2

A basis for PL functions is given by Courant functions, which take a value of 1 at a chosen vertex and 0 at all other vertices.



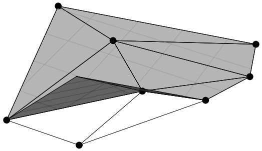
Tent Functions 2

A basis for PL functions is given by Courant functions, which take a value of 1 at a chosen vertex and 0 at all other vertices.



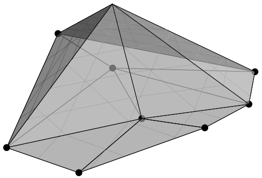
Tent Functions 2

A basis for PL functions is given by Courant functions, which take a value of 1 at a chosen vertex and 0 at all other vertices.



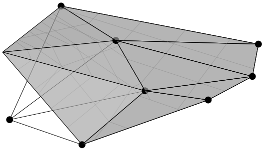
Tent Functions 2

A basis for PL functions is given by Courant functions, which take a value of 1 at a chosen vertex and 0 at all other vertices.



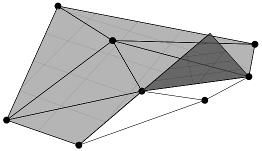
Tent Functions 2

A basis for PL functions is given by Courant functions, which take a value of 1 at a chosen vertex and 0 at all other vertices.



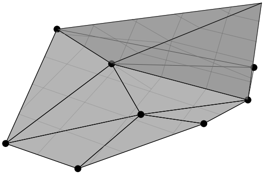
Tent Functions 2

A basis for PL functions is given by Courant functions, which take a value of 1 at a chosen vertex and 0 at all other vertices.



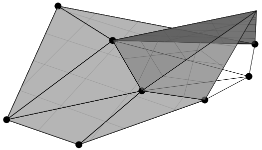
Tent Functions 2

A basis for PL functions is given by Courant functions, which take a value of 1 at a chosen vertex and 0 at all other vertices.



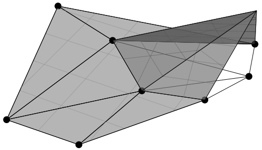
Tent Functions 2

A basis for PL functions is given by Courant functions, which take a value of 1 at a chosen vertex and 0 at all other vertices.



Tent Functions 2

A basis for PL functions is given by Courant functions, which take a value of 1 at a chosen vertex and 0 at all other vertices.



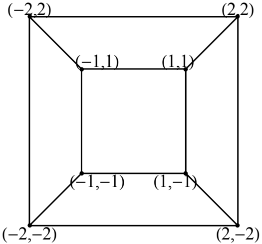
- Note: dimension only depends on number of vertices.
- No dependence on geometry!

Polygonal Subdivisions

What if we use a polygons instead of triangles?

Polygonal Subdivisions

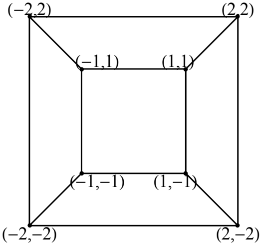
What if we use a polygons instead of triangles?



A polygonal subdivision

Polygonal Subdivisions

What if we use a polygons instead of triangles?

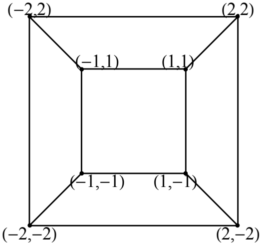


A polygonal subdivision

Does the dimension of the space of PL functions depend on geometry?

Polygonal Subdivisions

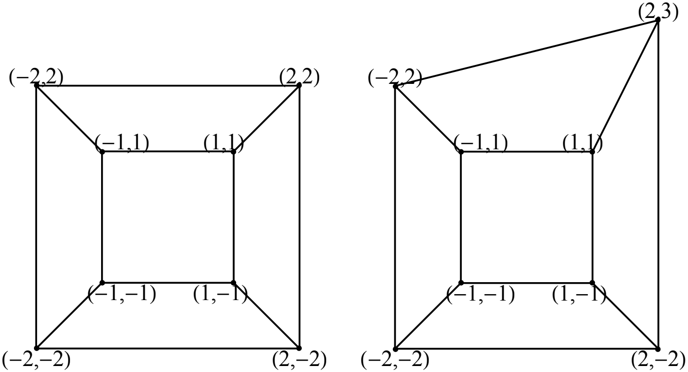
What if we use a polygons instead of triangles?



A polygonal subdivision

Does the dimension of the space of PL functions depend on geometry?
YES!

PL functions depend on geometry



Δ_1 $\qquad$ Δ_2

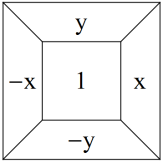
$\dim \mathrm{PL}$ functions $= 4$ $\qquad$ $\dim \mathrm{PL}$ functions $= 3$

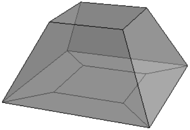
NonTrivial PL Functions

The graph of a PL function on Δ_1:

NonTrivial PL Functions

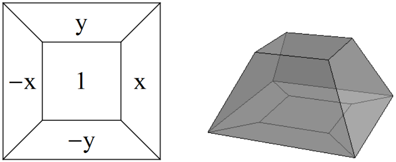
The graph of a PL function on Δ_1:





NonTrivial PL Functions

The graph of a PL function on Δ_1:



Δ_2 does not have this function!

Digression on polytopes

A d-**polytope** is a bounded intersection of half-spaces in $\mathbb{R}^d$.

Digression on polytopes

A d-**polytope** is a bounded intersection of half-spaces in $\mathbb{R}^d$.

Get a square by intersecting four half-spaces:

Digression on polytopes

A d-**polytope** is a bounded intersection of half-spaces in $\mathbb{R}^d$.

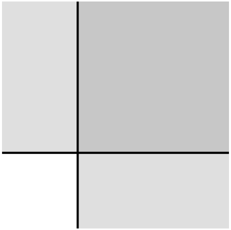
Get a square by intersecting four half-spaces:

$y \geq 0$

Digression on polytopes

A d-**polytope** is a bounded intersection of half-spaces in $\mathbb{R}^d$.

Get a square by intersecting four half-spaces:

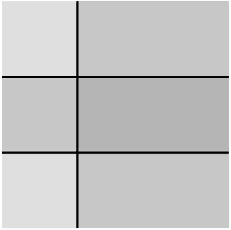


$$y \geq 0 \cap x \geq 0$$

Digression on polytopes

A d-**polytope** is a bounded intersection of half-spaces in $\mathbb{R}^d$.

Get a square by intersecting four half-spaces:

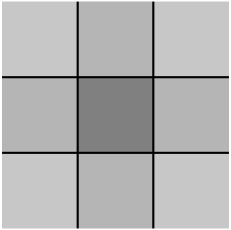


$y \geq 0 \cap x \geq 0 \cap y \leq 1$

Digression on polytopes

A d-**polytope** is a bounded intersection of half-spaces in $\mathbb{R}^d$.

Get a square by intersecting four half-spaces:



$$y \geq 0 \cap x \geq 0 \cap y \leq 1 \cap x \leq 1$$

Digression on polytopes

A d-**polytope** is a bounded intersection of half-spaces in $\mathbb{R}^d$.

Get a square by intersecting four half-spaces:

$$y \geq 0 \cap x \geq 0 \cap y \leq 1 \cap x \geq 1$$

Edge graphs

Polytopes have:

- vertices (0-dimensional),

Edge graphs

Polytopes have:

- vertices (0-dimensional),
- edges (1-dimensional)

Edge graphs

Polytopes have:

- vertices (0-dimensional),
- edges (1-dimensional)
- faces (2-dimensional)

Edge graphs

Polytopes have:

- vertices (0-dimensional),
- edges (1-dimensional)
- faces (2-dimensional)

$\vdots$

Edge graphs

Polytopes have:

- vertices (0-dimensional),
- edges (1-dimensional)
- faces (2-dimensional)

⋮

- i-faces (i-dimensional)

Edge graphs

Polytopes have:

- vertices (0-dimensional),
- edges (1-dimensional)
- faces (2-dimensional)

⋮

- i-faces (i-dimensional)

Edge graph of a polytope = graph formed by vertices and edges.

Square

Edge graphs

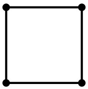
Polytopes have:

- vertices (0-dimensional),
- edges (1-dimensional)
- faces (2-dimensional)

$\vdots$

- i-faces (i-dimensional)

Edge graph of a polytope = graph formed by vertices and edges.



Edge graph of square

Planar graphs

Edge graphs of 3-polytopes are always **planar**.

Planar graphs

Edge graphs of 3-polytopes are always **planar**.

Graphs that can be drawn in the plane without crossing edges are called **planar graphs**.

Planar graphs

Edge graphs of 3-polytopes are always **planar**.

Graphs that can be drawn in the plane without crossing edges are called **planar graphs**.

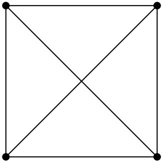
Complete graph on 4 vertices is planar:

Planar graphs

Edge graphs of 3-polytopes are always **planar**.

Graphs that can be drawn in the plane without crossing edges are called **planar graphs**.

Complete graph on 4 vertices is planar:

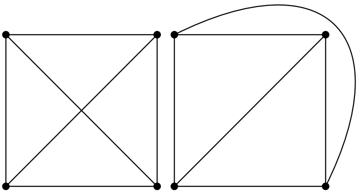


Planar graphs

Edge graphs of 3-polytopes are always **planar**.

Graphs that can be drawn in the plane without crossing edges are called **planar graphs**.

Complete graph on 4 vertices is planar:

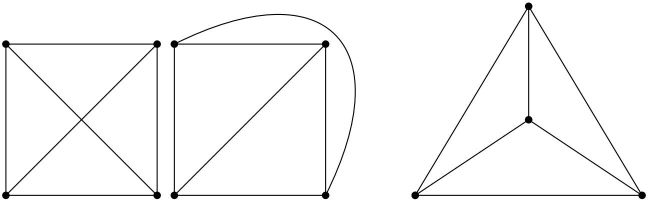


Planar graphs

Edge graphs of 3-polytopes are always **planar**.

Graphs that can be drawn in the plane without crossing edges are called **planar graphs**.

Complete graph on 4 vertices is planar:

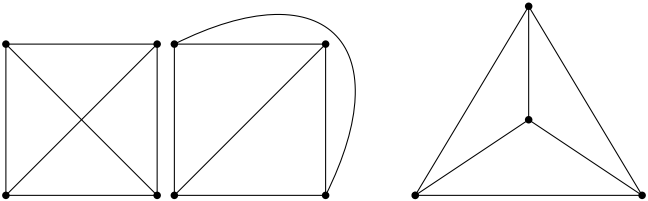


Planar graphs

Edge graphs of 3-polytopes are always **planar**.

Graphs that can be drawn in the plane without crossing edges are called **planar graphs**.

Complete graph on 4 vertices is planar:

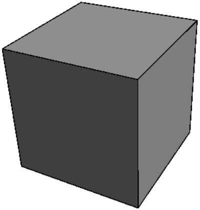


Edge graphs of 3-polytopes are planar because of **Schlegel diagrams** (edge shadow).

Schlegel Diagrams

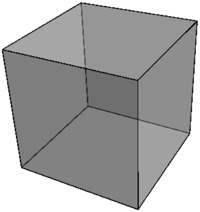
Schlegel Diagrams

Cube



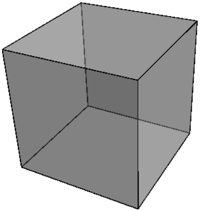
Schlegel Diagrams

Make it transparent

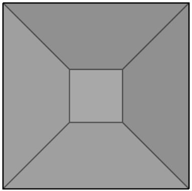


Schlegel Diagrams

Make it transparent

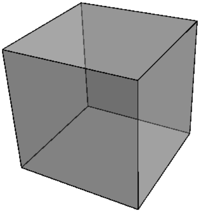


Look into one of the faces:

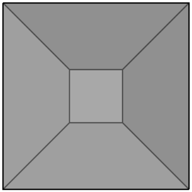


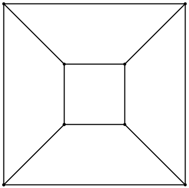
Schlegel Diagrams

Make it transparent



Look into one of the faces:

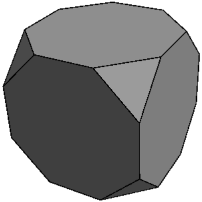




Schlegel diagram

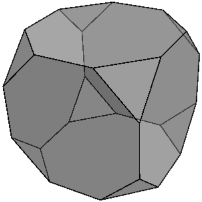
Schlegel Diagrams

Truncated cube



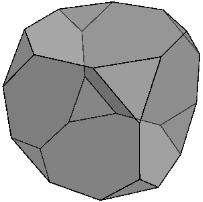
Schlegel Diagrams

Make it transparent

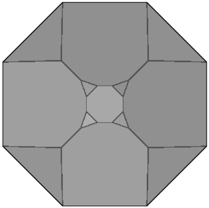


Schlegel Diagrams

Make it transparent

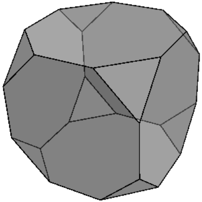


Look into an octagonal face:

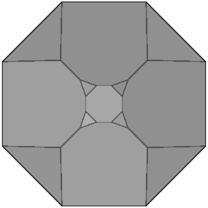


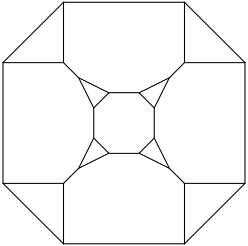
Schlegel Diagrams

Make it transparent



Look into an octagonal face:





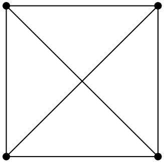
Schlegel diagram

Edge graphs of 3-polytopes

What planar graphs are edge graphs of 3-polytopes?

Edge graphs of 3-polytopes

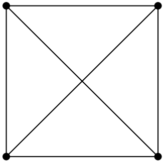
What planar graphs are edge graphs of 3-polytopes?



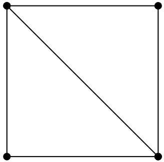
Edge graph
of 3-polytope

Edge graphs of 3-polytopes

What planar graphs are edge graphs of 3-polytopes?



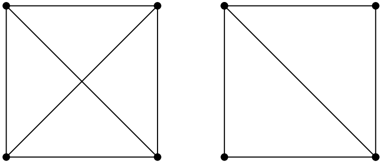
Edge graph
of 3-polytope



Not edge graph
of 3-polytope

Edge graphs of 3-polytopes

What planar graphs are edge graphs of 3-polytopes?



Edge graph of 3-polytope

Not edge graph of 3-polytope

Removing two vertices (and adjacent edges) can disconnect the graph.

Balinski's Theorem

A graph is d-**connected** if removing $(d-1)$ vertices does not disconnect the graph.

Balinski's Theorem

A graph is d-**connected** if removing $(d-1)$ vertices does not disconnect the graph.

Balinski's Theorem (1961)

The edge graph of a d-polytope is d-connected.

Balinski's Theorem

A graph is d-**connected** if removing $(d-1)$ vertices does not disconnect the graph.

Balinski's Theorem (1961)

The edge graph of a d-polytope is d-connected.

The edge graph of a cube is 3-connected:

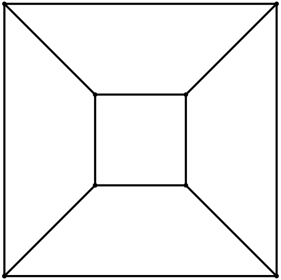
Balinski's Theorem

A graph is d-**connected** if removing $(d-1)$ vertices does not disconnect the graph.

Balinski's Theorem (1961)

The edge graph of a d-polytope is d-connected.

The edge graph of a cube is 3-connected:



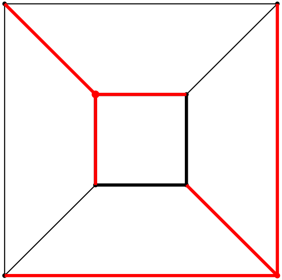
Balinski's Theorem

A graph is d-**connected** if removing $(d-1)$ vertices does not disconnect the graph.

Balinski's Theorem (1961)

The edge graph of a d-polytope is d-connected.

The edge graph of a cube is 3-connected:



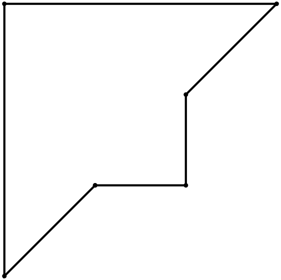
Balinski's Theorem

A graph is d-**connected** if removing $(d-1)$ vertices does not disconnect the graph.

Balinski's Theorem (1961)

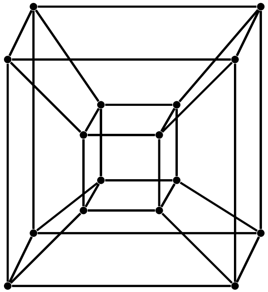
The edge graph of a d-polytope is d-connected.

The edge graph of a cube is 3-connected:



Balinski's Theorem

The edge graph of a 4-dimensional cube is 4-connected:



Steinitz' Theorem

Steinitz' Theorem

A graph is the edge graph of a 3-polytope if and only if the graph is planar and 3-connected.

Steinitz' Theorem

Steinitz' Theorem

A graph is the edge graph of a 3-polytope if and only if the graph is planar and 3-connected.

If a graph is planar and 3-connected,

Steinitz' Theorem

Steinitz' Theorem

A graph is the edge graph of a 3-polytope if and only if the graph is planar and 3-connected.

If a graph is planar and 3-connected,

- Is it possible to draw the graph without edges crossing so that the edges are all straight?

Steinitz' Theorem

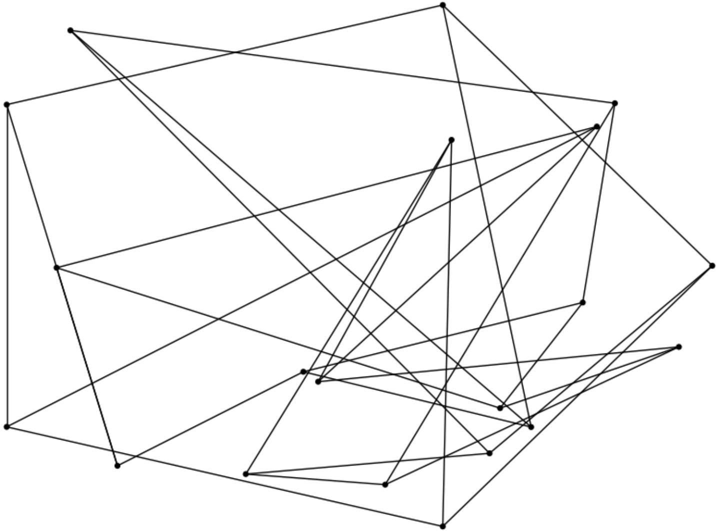
Steinitz' Theorem

A graph is the edge graph of a 3-polytope if and only if the graph is planar and 3-connected.

If a graph is planar and 3-connected,

- Is it possible to draw the graph without edges crossing so that the edges are all straight?
- Can you identify polytope for which the graph is the edge polytope?

A planar 3-connected graph



Tutte's embedding

Tutte's idea (1960) - given a planar 3-connected graph (drawn in any way):

Tutte's embedding

Tutte's idea (1960) - given a planar 3-connected graph (drawn in any way):

- Fix a cycle C which satisfies that the only edges between vertices of C are edges of C (no 'chords')

Tutte's embedding

Tutte's idea (1960) - given a planar 3-connected graph (drawn in any way):

- Fix a cycle C which satisfies that the only edges between vertices of C are edges of C (no 'chords')
- Edges in C are not movable

Tutte's embedding

Tutte's idea (1960) - given a planar 3-connected graph (drawn in any way):

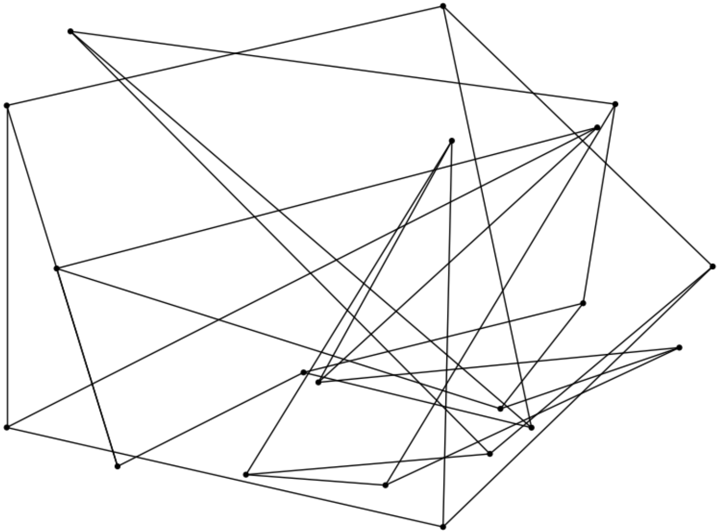
- Fix a cycle C which satisfies that the only edges between vertices of C are edges of C (no 'chords')
- Edges in C are not movable
- Edges not in C="rubber bands"

Tutte's embedding

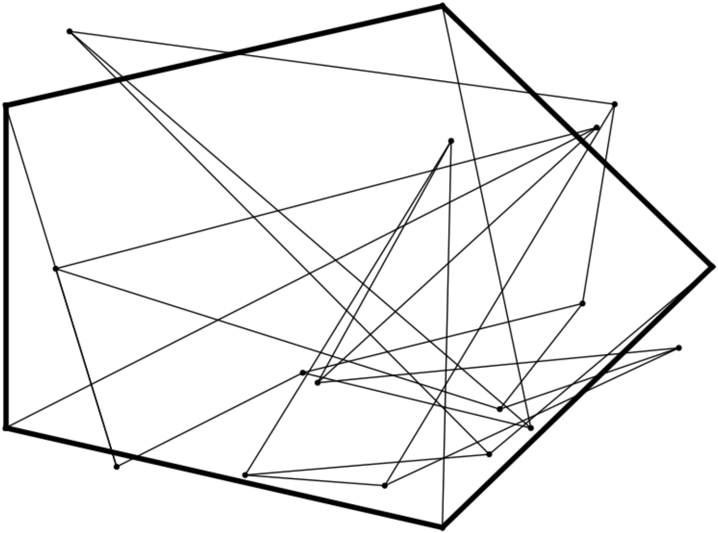
Tutte's idea (1960) - given a planar 3-connected graph (drawn in any way):

- Fix a cycle C which satisfies that the only edges between vertices of C are edges of C (no 'chords')
- Edges in C are not movable
- Edges not in C= "rubber bands"
- Let go! Then...

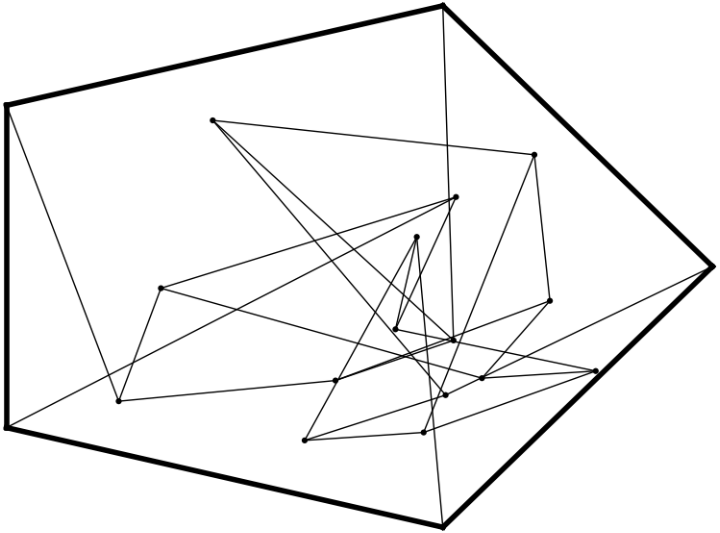
Tutte's embedding, continued



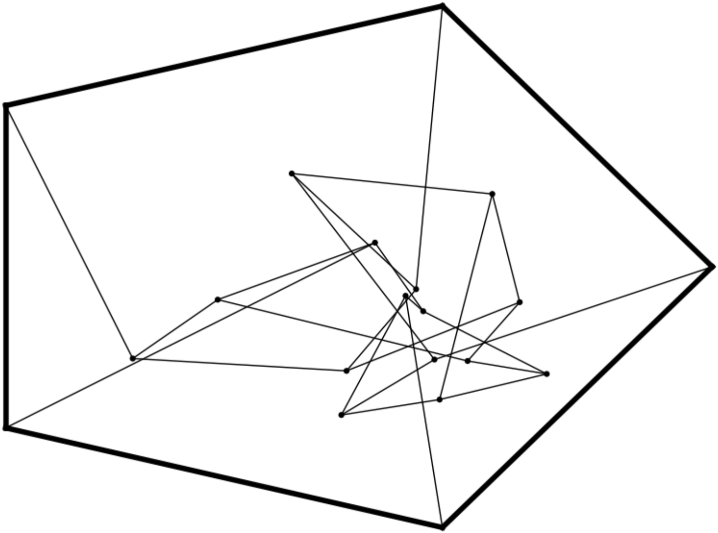
Tutte's embedding, continued



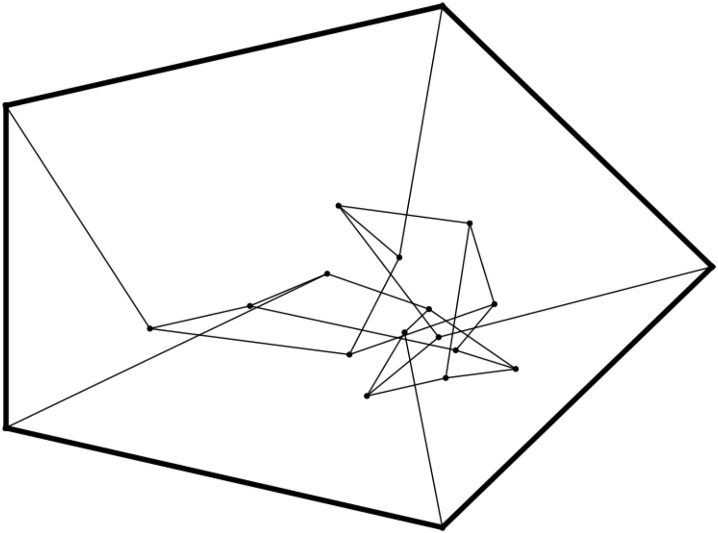
Tutte's embedding, continued



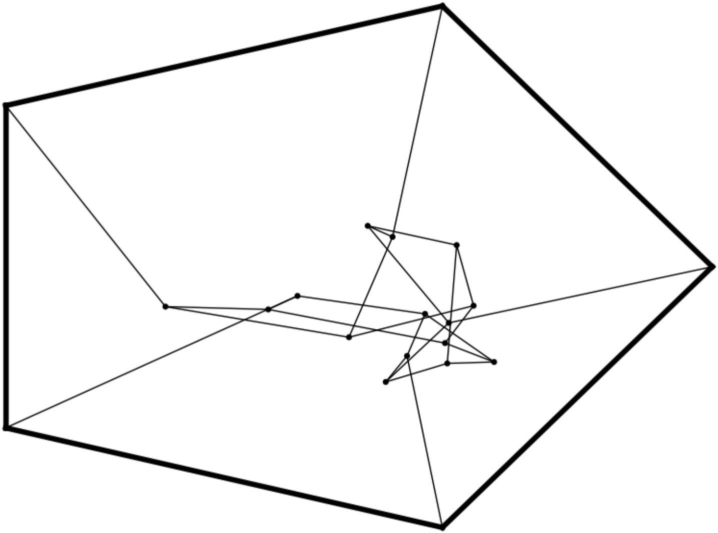
Tutte's embedding, continued



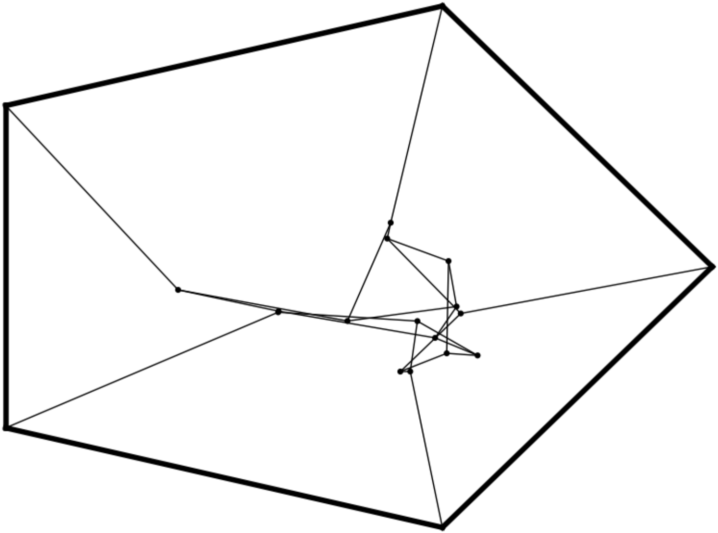
Tutte's embedding, continued



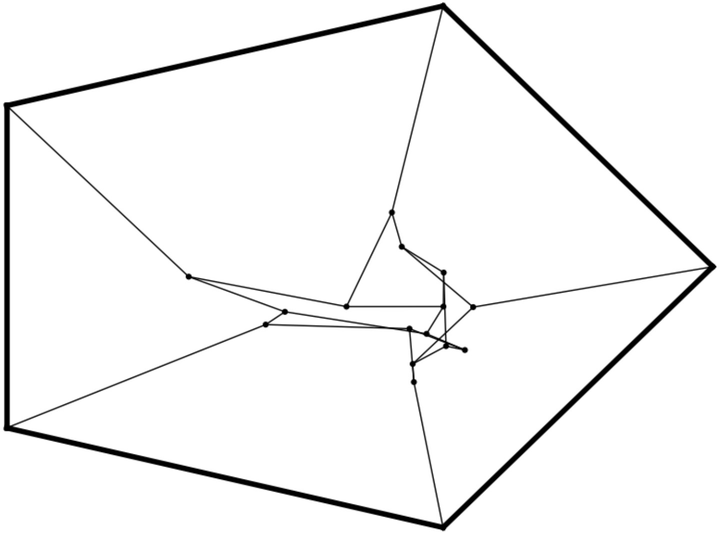
Tutte's embedding, continued



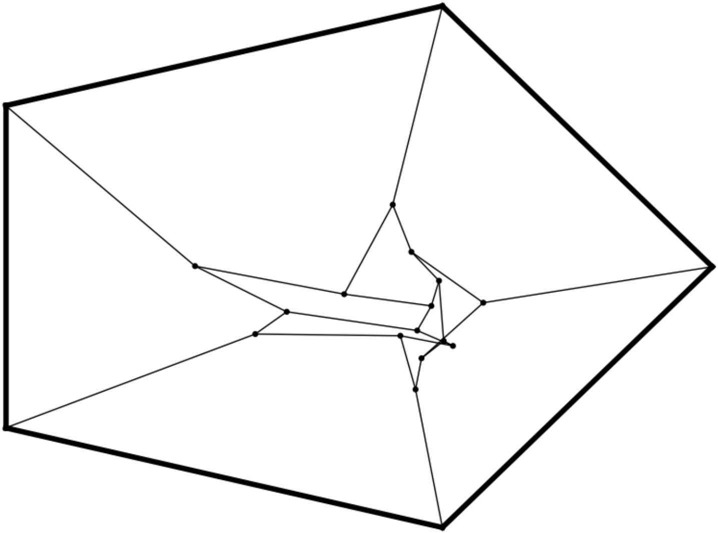
Tutte's embedding, continued



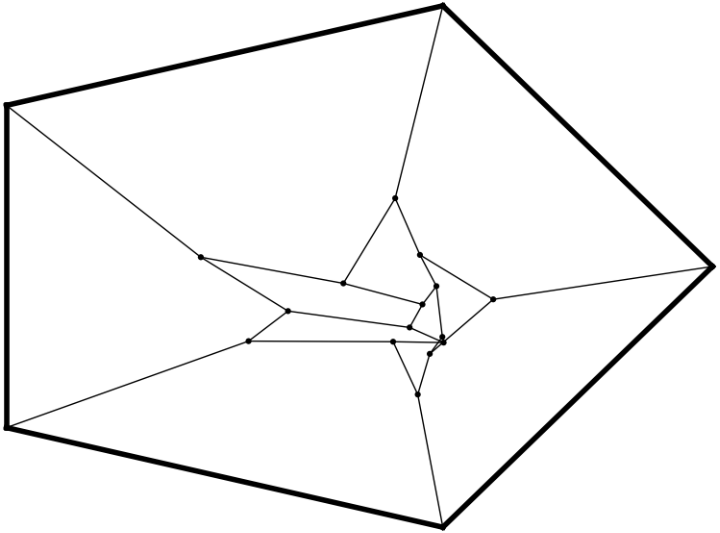
Tutte's embedding, continued



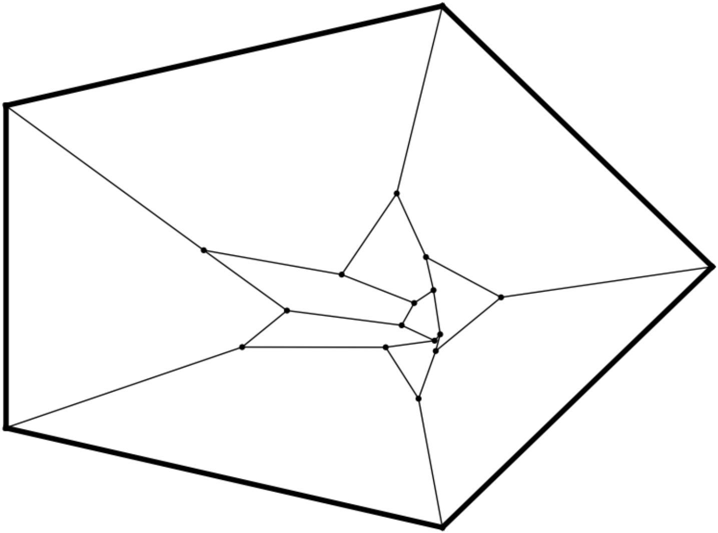
Tutte's embedding, continued



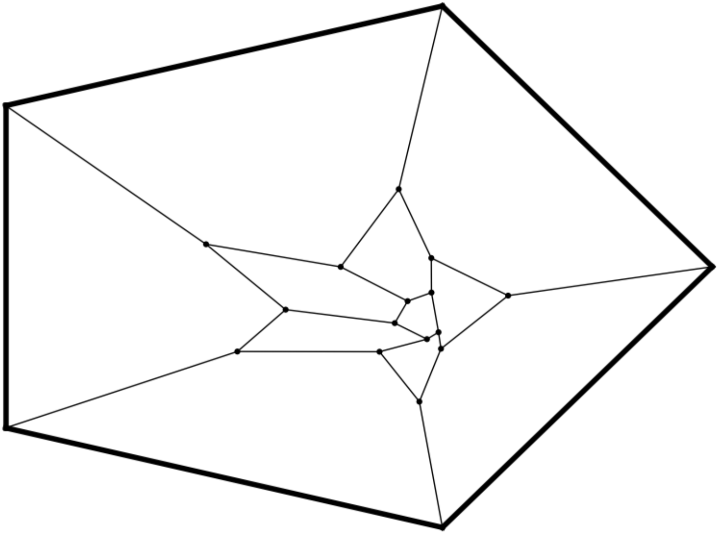
Tutte's embedding, continued



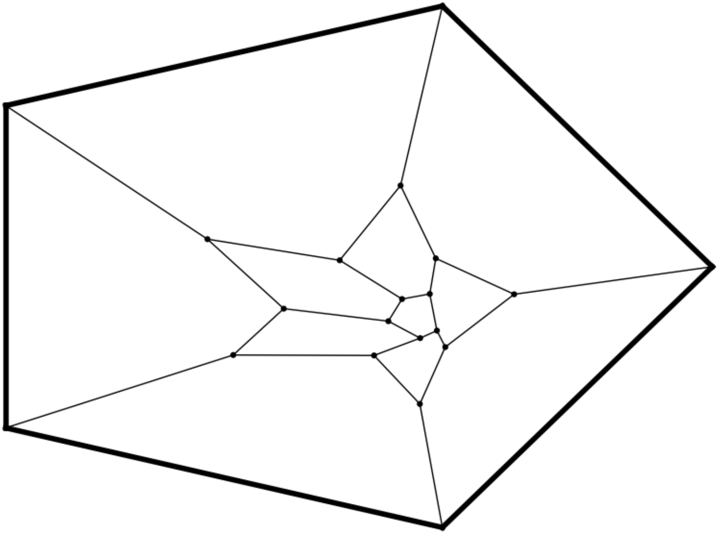
Tutte's embedding, continued



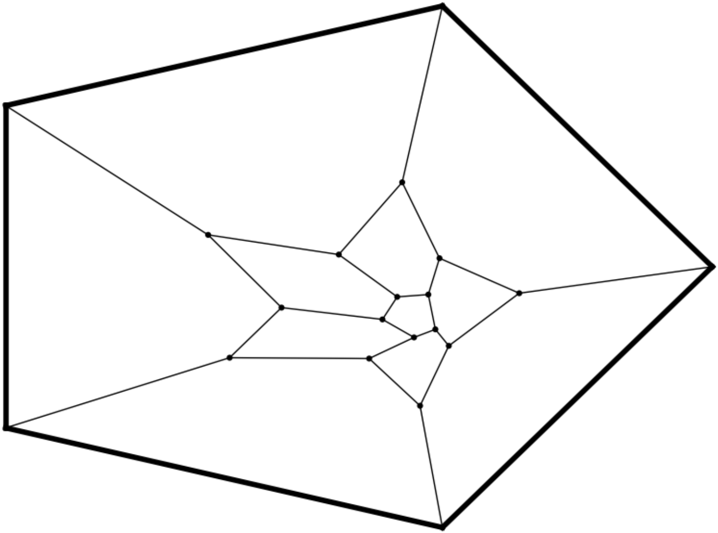
Tutte's embedding, continued



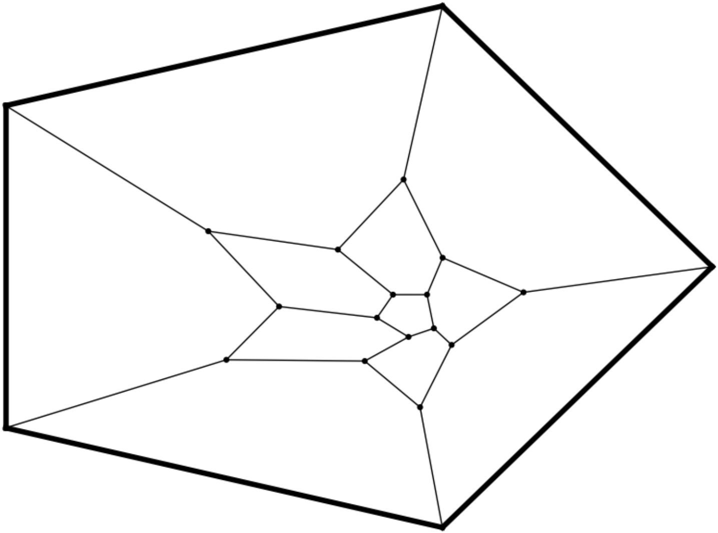
Tutte's embedding, continued



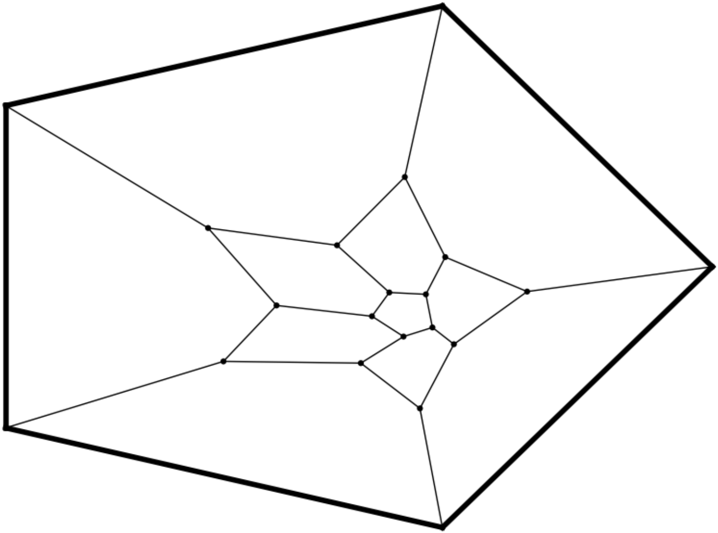
Tutte's embedding, continued



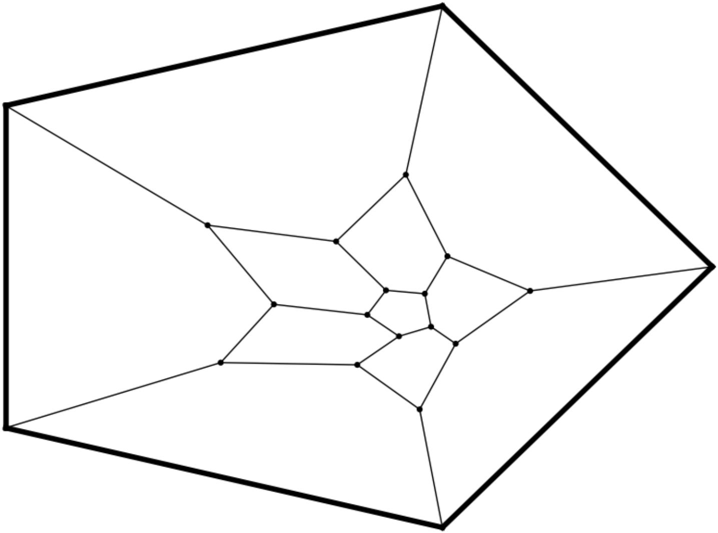
Tutte's embedding, continued



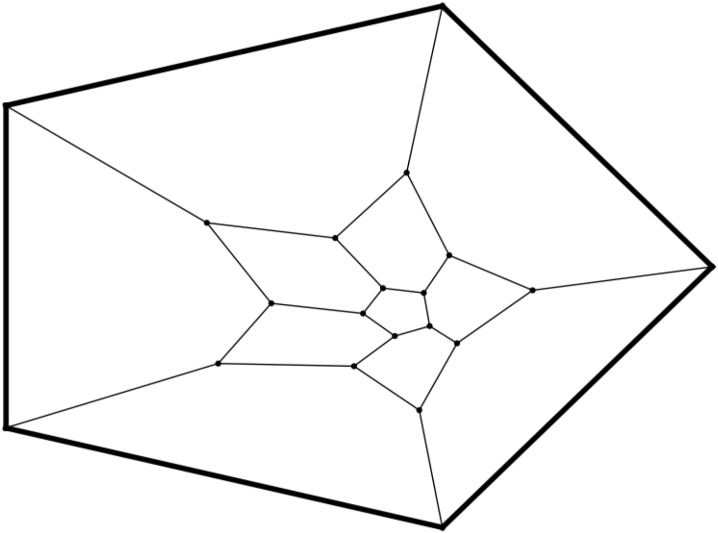
Tutte's embedding, continued



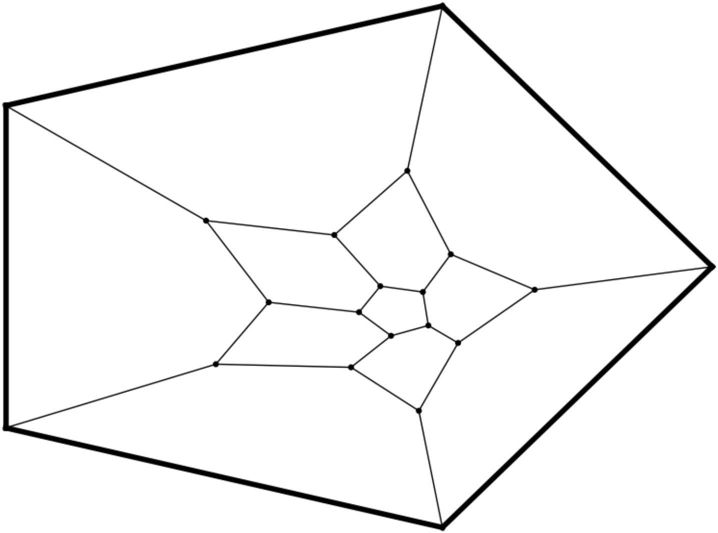
Tutte's embedding, continued



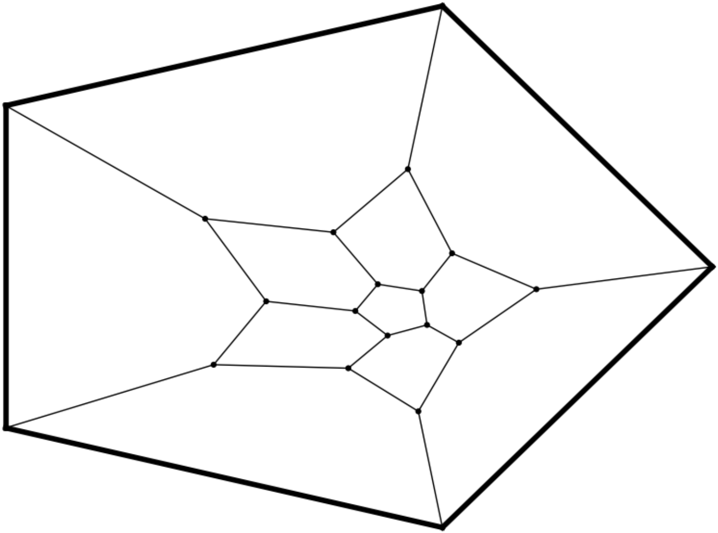
Tutte's embedding, continued



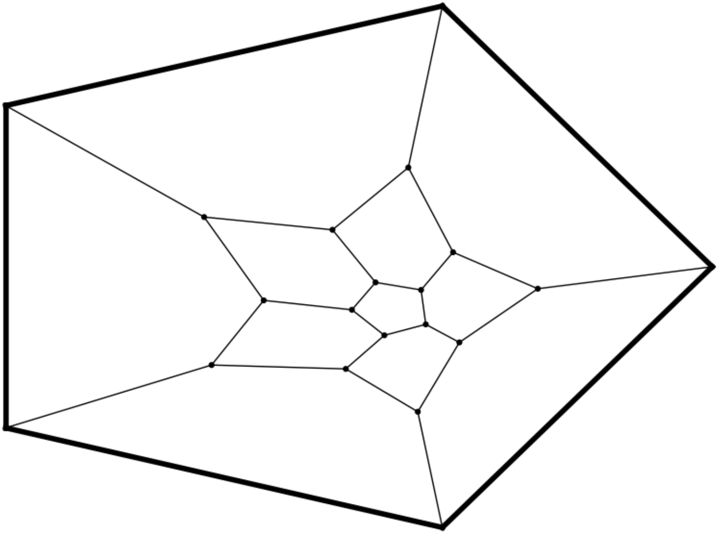
Tutte's embedding, continued



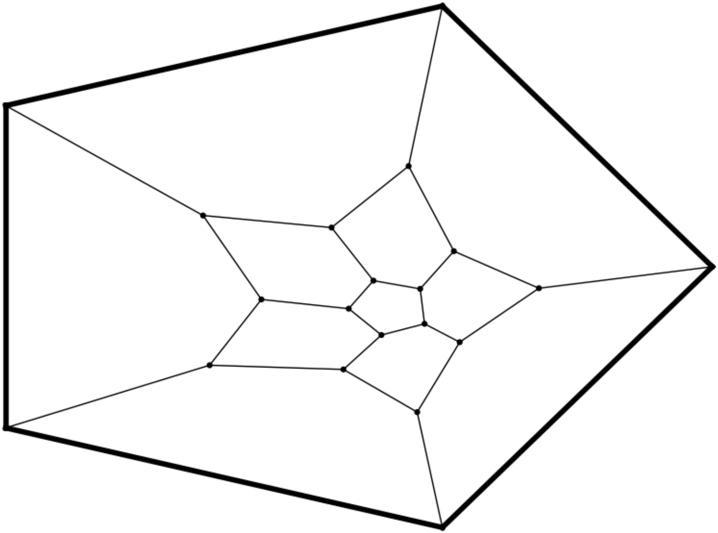
Tutte's embedding, continued



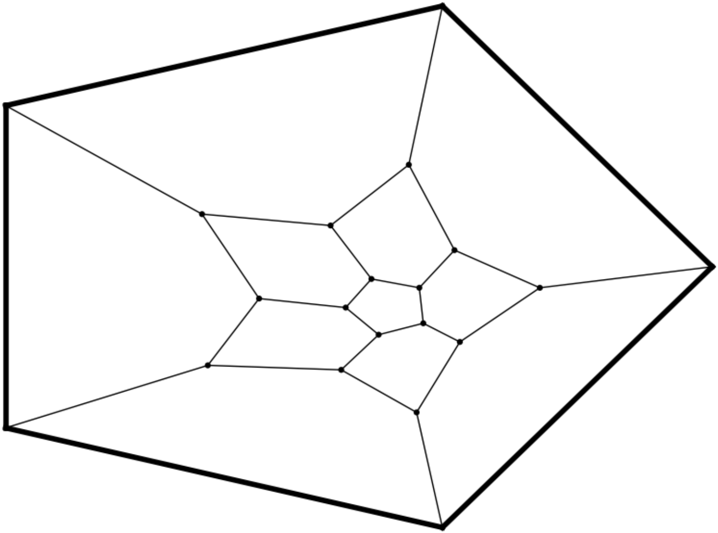
Tutte's embedding, continued



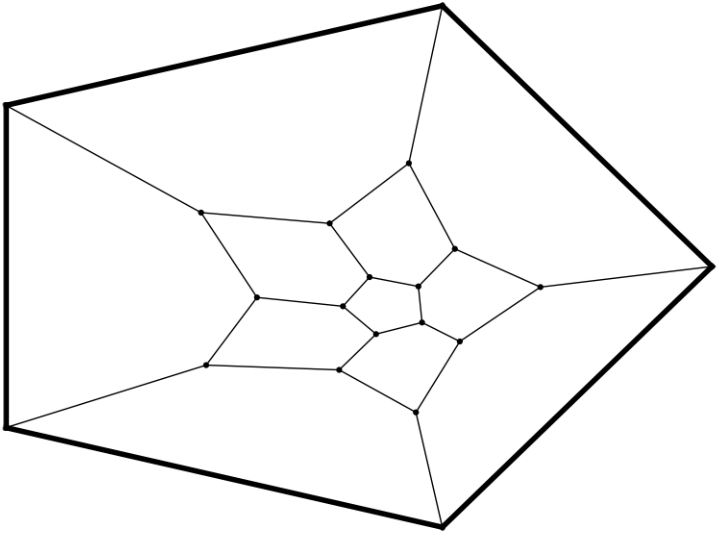
Tutte's embedding, continued



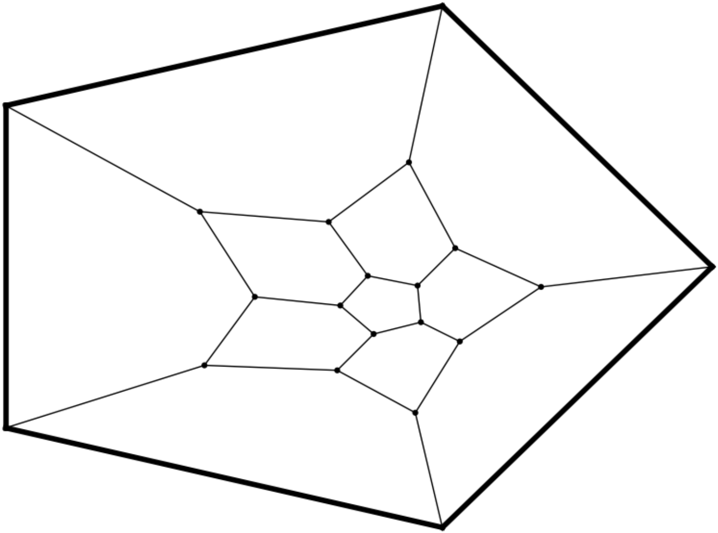
Tutte's embedding, continued



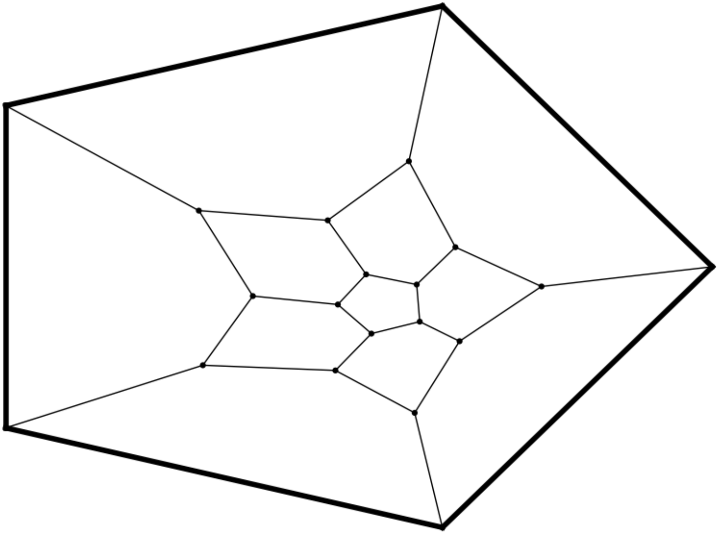
Tutte's embedding, continued



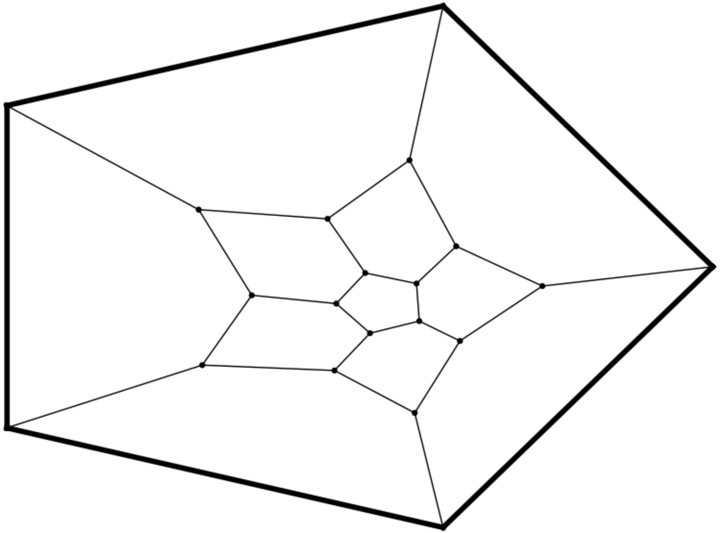
Tutte's embedding, continued



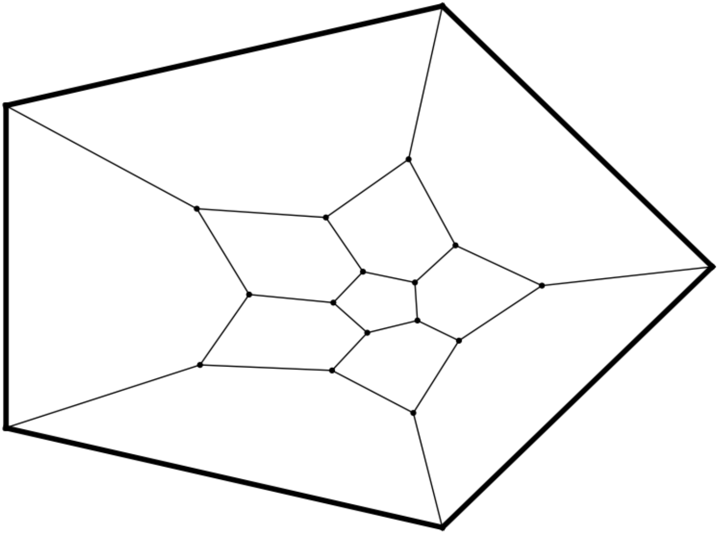
Tutte's embedding, continued



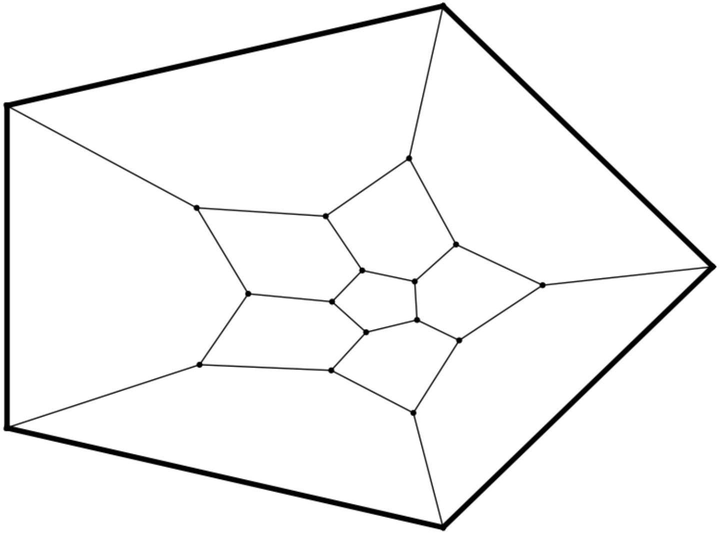
Tutte's embedding, continued



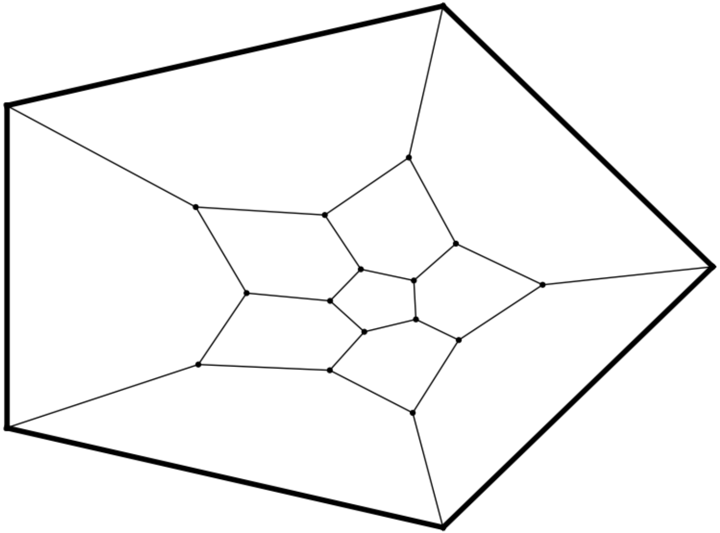
Tutte's embedding, continued



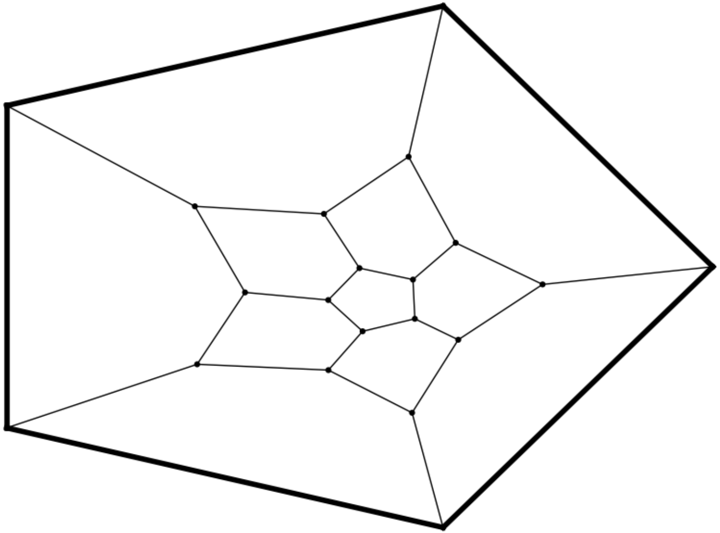
Tutte's embedding, continued



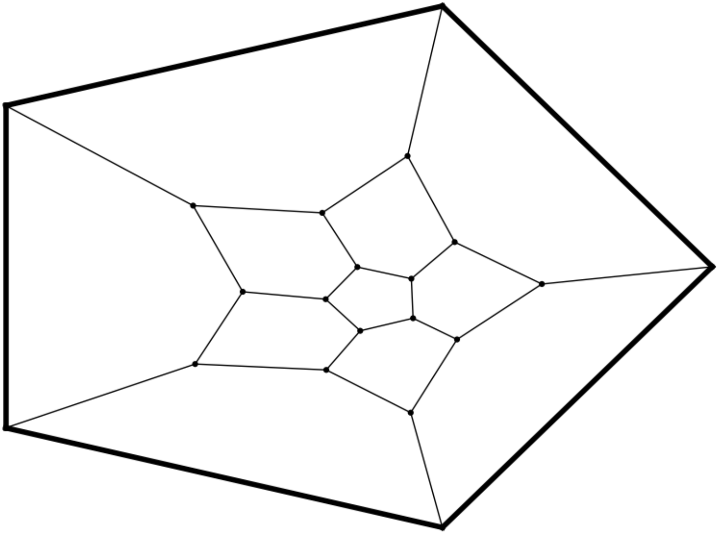
Tutte's embedding, continued



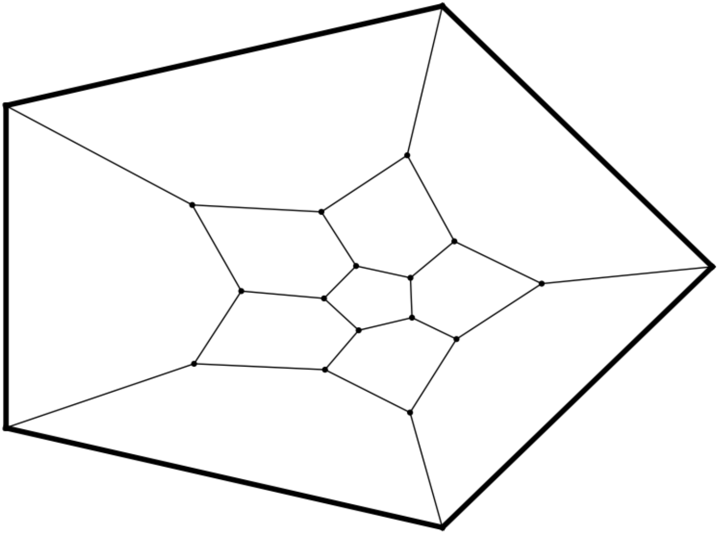
Tutte's embedding, continued



Tutte's embedding, continued



Tutte's embedding, continued



Tutte's embedding, continued

Graph will stabilize at a drawing for which

Tutte's embedding, continued

Graph will stabilize at a drawing for which

- C is the 'outer cycle',

Tutte's embedding, continued

Graph will stabilize at a drawing for which

- C is the 'outer cycle',
- no edges cross (Tutte 1960),

Tutte's embedding, continued

Graph will stabilize at a drawing for which

- C is the 'outer cycle',
- no edges cross (Tutte 1960),
- all polygonal faces are convex (Tutte 1960),

Tutte's embedding, continued

Graph will stabilize at a drawing for which

- C is the 'outer cycle',
- no edges cross (Tutte 1960),
- all polygonal faces are convex (Tutte 1960),
- the drawing is a **vertical projection** of the graph of a PL function! (Crapo and Whiteley 1982,1993)

Tutte's embedding, continued

Graph will stabilize at a drawing for which

- C is the 'outer cycle',
- no edges cross (Tutte 1960),
- all polygonal faces are convex (Tutte 1960),
- the drawing is a **vertical projection** of the graph of a PL function! (Crapo and Whiteley 1982,1993)

Tutte's idea inspired many methods which are widely used in geometric modeling and computer graphics.

Unpacking Tutte's embedding

Tutte's embedding is the **minimum** of an energy function:

Unpacking Tutte's embedding

Tutte's embedding is the **minimum** of an energy function:

$$F(V) = \sum_{\{i,j\} \in E, \{i,j\} \notin C} \omega_{ij} ||\mathbf{v}_i - \mathbf{v}_j||^2,$$

Unpacking Tutte's embedding

Tutte's embedding is the **minimum** of an energy function:

$$F(V) = \sum_{\{i,j\} \in E, \{i,j\} \notin C} \omega_{ij} ||\mathbf{v}_i - \mathbf{v}_j||^2,$$

- $\mathbf{v}_i$ is the vector of coordinates of vertex v_i and

Unpacking Tutte's embedding

Tutte's embedding is the **minimum** of an energy function:

$$F(V) = \sum_{\{i,j\} \in E, \{i,j\} \notin C} \omega_{ij} ||\mathbf{v}_i - \mathbf{v}_j||^2,$$

- $\mathbf{v}_i$ is the vector of coordinates of vertex v_i and
- the constants ω_{ij} indicate 'strength' of the rubber bands.

Unpacking Tutte's embedding

Tutte's embedding is the **minimum** of an energy function:

$$F(V) = \sum_{\{i,j\}\in E, \{i,j\}\notin C} \omega_{ij} ||\mathbf{v}_i - \mathbf{v}_j||^2,$$

- $\mathbf{v}_i$ is the vector of coordinates of vertex v_i and
- the constants ω_{ij} indicate 'strength' of the rubber bands.

- To find local minima, set $\nabla F = \mathbf{0}$.

Unpacking Tutte's embedding

Tutte's embedding is the **minimum** of an energy function:

$$F(V) = \sum_{\{i,j\} \in E, \{i,j\} \notin C} \omega_{ij} ||\mathbf{v}_i - \mathbf{v}_j||^2,$$

- $\mathbf{v}_i$ is the vector of coordinates of vertex v_i and
- the constants ω_{ij} indicate 'strength' of the rubber bands.

- To find local minima, set $\nabla F = \mathbf{0}$.
- Yields a **force balancing** equation at each vertex:

Unpacking Tutte's embedding

Tutte's embedding is the **minimum** of an energy function:

$$F(V) = \sum_{\{i,j\}\in E, \{i,j\}\notin C} \omega_{ij} ||\mathbf{v}_i - \mathbf{v}_j||^2,$$

- $\mathbf{v}_i$ is the vector of coordinates of vertex v_i and
- the constants ω_{ij} indicate 'strength' of the rubber bands.

- To find local minima, set $\nabla F = \mathbf{0}$.
- Yields a **force balancing** equation at each vertex:

$$\sum_{v_j \text{ adjacent to } v_i} \omega_{ij}(\mathbf{v}_j - \mathbf{v}_i) = 0.$$

Unpacking Tutte's embedding

Tutte's embedding is the **minimum** of an energy function:

$$F(V) = \sum_{\{i,j\}\in E, \{i,j\}\notin C} \omega_{ij} ||\mathbf{v}_i - \mathbf{v}_j||^2,$$

- $\mathbf{v}_i$ is the vector of coordinates of vertex v_i and
- the constants ω_{ij} indicate 'strength' of the rubber bands.

- To find local minima, set $\nabla F = \mathbf{0}$.
- Yields a **force balancing** equation at each vertex:

$$\sum_{v_j \text{ adjacent to } v_i} \omega_{ij}(\mathbf{v}_j - \mathbf{v}_i) = 0.$$

Forces balance $\implies$ vertices don't move anymore!

Self-Stress

Self-Stress

A collection of constants ω_{ij} for each edge $\{i, j\}$ in a graph with vertex coordinates $\mathbf{v}_i$ satisfying the force balancing equations

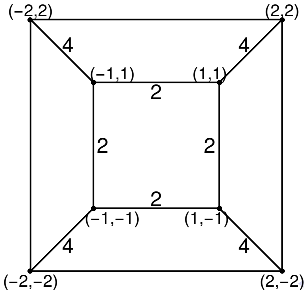
$$\sum_{v_j \text{ adjacent to } v_i} \omega_{ij}(\mathbf{v}_j - \mathbf{v}_i) = 0.$$

Self-Stress

Self-Stress

A collection of constants ω_{ij} for each edge $\{i, j\}$ in a graph with vertex coordinates $\mathbf{v}_i$ satisfying the force balancing equations

$$\sum_{v_j \text{ adjacent to } v_i} \omega_{ij}(\mathbf{v}_j - \mathbf{v}_i) = 0.$$

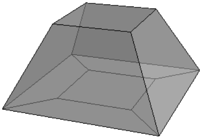


Maxwell's Observation (1860s)

Self-stresses are in 1-1 correspondence (almost) with nontrivial PL functions!

Maxwell's Observation (1860s)

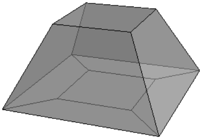
Self-stresses are in 1-1 correspondence (almost) with nontrivial PL functions!



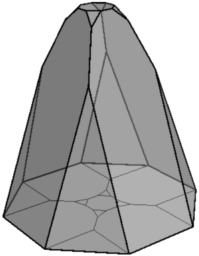
From self-stress
on last slide

Maxwell's Observation (1860s)

Self-stresses are in 1-1 correspondence (almost) with nontrivial PL functions!



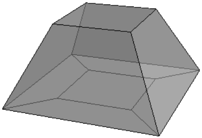
From self-stress on last slide



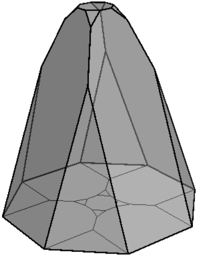
PL function from self-stress on Schlegel diagram of truncated cube

Maxwell's Observation (1860s)

Self-stresses are in 1-1 correspondence (almost) with nontrivial PL functions!



From self-stress on last slide



PL function from self-stress on Schlegel diagram of truncated cube

Computing dimension of PL functions
=computing space of self-stresses

Where to now?

Numerical analysis

Where to now?

Numerical analysis

- Compute dimension formulas for higher degree piecewise polynomials (splines) over meshes which have higher order derivatives

Where to now?

Numerical analysis

- Compute dimension formulas for higher degree piecewise polynomials (splines) over meshes which have higher order derivatives
- Useful for Finite Element Method in partial differential equations

Where to now?

Numerical analysis

- Compute dimension formulas for higher degree piecewise polynomials (splines) over meshes which have higher order derivatives
- Useful for Finite Element Method in partial differential equations
- Connections to algebraic geometry, commutative algebra, homological algebra

Where to now?

Numerical analysis

- Compute dimension formulas for higher degree piecewise polynomials (splines) over meshes which have higher order derivatives
- Useful for Finite Element Method in partial differential equations
- Connections to algebraic geometry, commutative algebra, homological algebra
- Relates to problem of determining the dimension of the space of polynomials which vanish to a fixed order on a set of points (algebraic geometry)

Where to now?

Numerical analysis

- Compute dimension formulas for higher degree piecewise polynomials (splines) over meshes which have higher order derivatives
- Useful for Finite Element Method in partial differential equations
- Connections to algebraic geometry, commutative algebra, homological algebra
- Relates to problem of determining the dimension of the space of polynomials which vanish to a fixed order on a set of points (algebraic geometry)
- Famous conjecture of Nagata related to Hilbert's fourteenth problem

Where to now?

Rigidity Theory

Where to now?

Rigidity Theory

(Dates back to Cauchy's rigidity theorem for convex polytopes)

Where to now?

Rigidity Theory

(Dates back to Cauchy's rigidity theorem for convex polytopes)

- Infinitesimal motions of bar and joint frameworks are **dual** to self-stresses

Where to now?

Rigidity Theory

(Dates back to Cauchy's rigidity theorem for convex polytopes)

- Infinitesimal motions of bar and joint frameworks are **dual** to self-stresses
- Engineering applications - important to determine stability of structures

Where to now?

Rigidity Theory

(Dates back to Cauchy's rigidity theorem for convex polytopes)

- Infinitesimal motions of bar and joint frameworks are **dual** to self-stresses
- Engineering applications - important to determine stability of structures
- Algebraic combinatorics - the g-theorem (Stanley 1980, Billera and Lee 1981) for simplicial polytopes relies on ideas from rigidity

Where to now?

Rigidity Theory

(Dates back to Cauchy's rigidity theorem for convex polytopes)

- Infinitesimal motions of bar and joint frameworks are **dual** to self-stresses
- Engineering applications - important to determine stability of structures
- Algebraic combinatorics - the g-theorem (Stanley 1980, Billera and Lee 1981) for simplicial polytopes relies on ideas from rigidity

The list goes on...

THANK YOU!