Two Tales of Freeness

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Two Algebraic Objects

- Pure n-dimensional polytopal complex $\Delta \subset \mathbb{R}^n$ (subdivision of region homeomorphic to n-dimensional ball by convex polytopes)
- Module $C^0(\Delta)$ of continuous functions piecewise polynomial with respect to Δ

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- Module $C^0(\Delta)$ of continuous functions piecewise polynomial with respect to Δ

- Hyperplane arrangement $\mathcal{A} \subset \mathbb{K}^n$ ($\mathbb{K}$ a field) (union of hyperplanes in $\mathbb{K}^n$)
- Module $D(\mathcal{A})$ of vector fields tangent to $\mathcal{A}$

Algebraic structure (in particular, *freeness*) of $C^0(\Delta)$ and $D(\mathcal{A})$ depend on

- Combinatorics of Δ (number of faces of dimension i) and $\mathcal{A}$ (*intersection lattice* of $\mathcal{A}$)
- Geometry of $\Delta, \mathcal{A}$ (how each is embedded in ambient space)

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We'll discuss

- What is freeness? (contextually)
- Why should anyone care? (implications of freeness)
- What connections are there between $D(\mathcal{A})$ and $C^0(\Delta)$? What light do these shed on freeness in the two contexts?

Part I: Continuous Splines

Continuous Piecewise Polynomials

Continuous Spline

A continuous piecewise polynomial function.

Notation:

- $C^0(\Delta)$ = continuous piecewise polynomial functions over a subdivision Δ
- $C^0_d(\Delta)$ = $\mathbb{R}$-vector space of splines whose restriction to each polytope is a polynomial of degree $\leq d$

Main problem: Compute $\dim C^0_d(\Delta)$.

Some Context: Splines in Calculus 1

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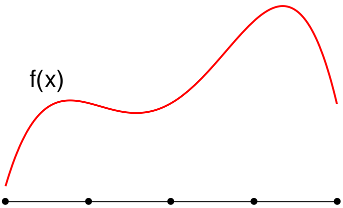

Graph of a function

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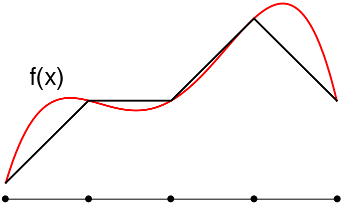

Trapezoid Rule

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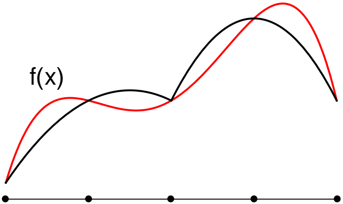

Simpson's Rule

Two Subintervals

$\Delta = [a_0, a_1] \cup [a_1, a_2]$ (assume WLOG $a_1 = 0$)

$$\begin{aligned}(f_1, f_2) \in C^0_d(\Delta) &\iff f_1(0) = f_2(0)\\ &\iff x | (f_2 - f_1)\end{aligned}$$

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Even more explicitly:

- $f_1(x) = b_0 + b_1 x + \cdots + b_d x^d$
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$$\dim C^0_d(\Delta) = 2d + 1 \text{ for } d \geq 0$$

Higher Dimensions

More General Problem: Compute $\dim C^0_d(\Delta)$ where $\Delta \subset \mathbb{R}^n$

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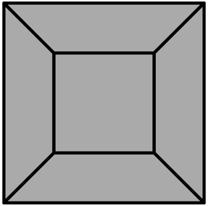

A polytopal complex

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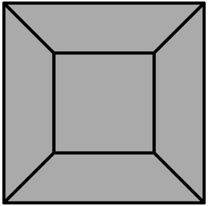
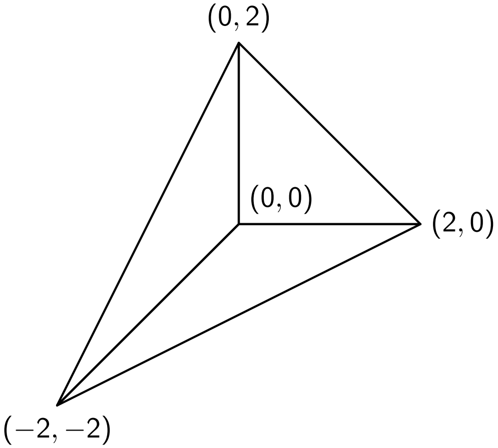


A polytopal complex

(Algebraic) Spline Criterion:

- If $\tau \in \Delta_{n-1}$, l_τ = affine form vanishing on affine span of τ
- Collection $\{f_\sigma\}_{\sigma \in \Delta_n}$ glue to $F \in C^0(\Delta) \iff$ for every pair of adjacent facets $\sigma_1, \sigma_2 \in \Delta_n$ with $\sigma_1 \cap \sigma_2 = \tau \in \Delta_{n-1}$, $l_\tau | (f_{\sigma_1} - f_{\sigma_2})$

Continuous Splines in Two Dimensions



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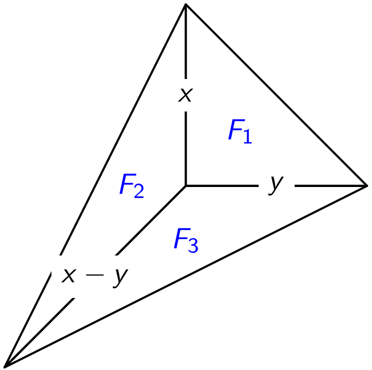

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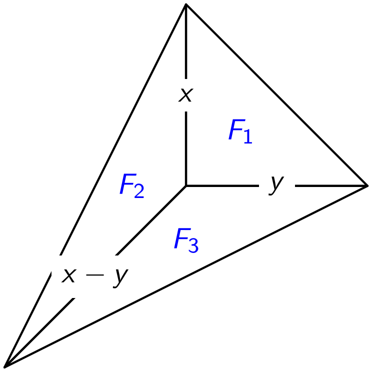
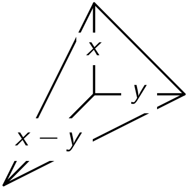


$(F_1, F_2, F_3) \in C^0(\Delta) \iff$
$\exists f_1, f_2, f_3$ so that

$$\begin{aligned} F_1 - F_2 &= f_1 x \\ F_2 - F_3 &= f_2(x-y) \\ F_3 - F_1 &= f_3 y \end{aligned}$$

Freeness



Three splines in $C^0(\Delta)$:

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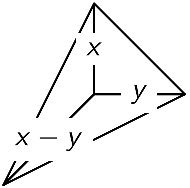

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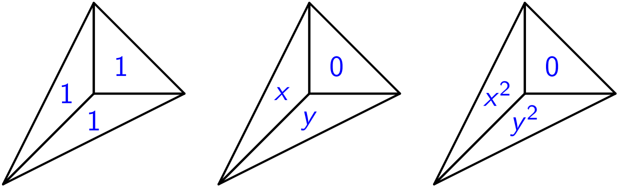

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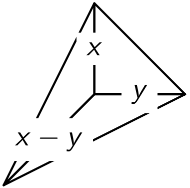

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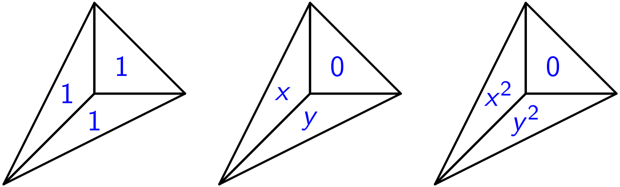

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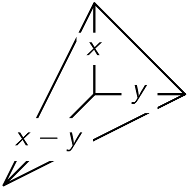
Freeness



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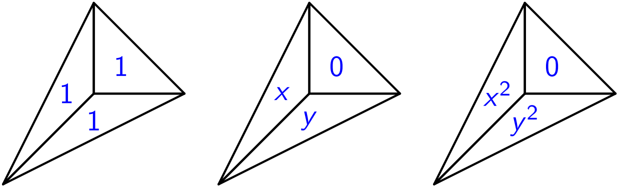

- In fact, every spline $F \in C^0(\Delta)$ can be written uniquely as a polynomial combination of these three splines.
- We say $C^0(\Delta)$ is a **free** $\mathbb{R}[x, y]$-module, generated in **degrees** $0, 1, 2$

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- $C^0(\Delta)_d \cong \mathbb{R}[x,y]_d(1,1,1) \oplus \mathbb{R}[x,y]_{d-1}(0,x,y) \oplus \mathbb{R}[x,y]_{d-2}(0,x^2,y^2)$.
- $\dim C^0(\Delta)_d = (d+1) + (d) + (d-1) = 3d$ for $d \geq 1$.

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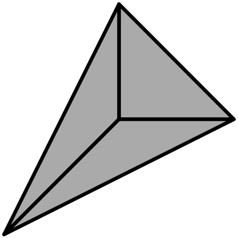
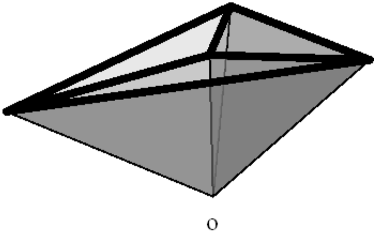
In general, employ a *coning* construction $\Delta \to \widehat{\Delta}$ to homogenize and consider $\dim C^0(\widehat{\Delta})_d$.

Coning Construction

- $\widehat{\Delta} \subset \mathbb{R}^{n+1}$ denotes the cone over $\Delta \subset \mathbb{R}^n$.



Δ



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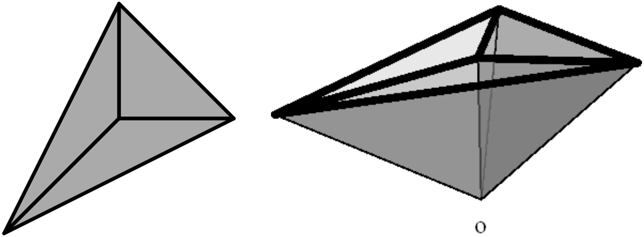

Δ $\widehat{\Delta}$

- $C^0(\widehat{\Delta})$ is always a **graded** module over $\mathbb{R}[x_0, \ldots, x_n]$
- $C^0_d(\Delta) \cong C^0(\widehat{\Delta})_d$ [Billera-Rose '91]

Consequences of Freeness

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- Freeness of $C^0(\widehat{\Delta})$ is highly studied:
  - via localization [Billera-Rose '92]
  - via sheaves on posets [Yuzvinsky '92]
  - via dual graphs [Rose '95]
  - via homologies of a chain complex [Schenck '97] (Δ simplicial)

C^0 for triangulations: Courant functions

A basis for $C^0_1(\Delta)$ is given by Courant functions T_v, which take a value of 1 at a chosen vertex v and 0 at all other vertices.

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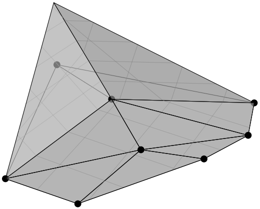

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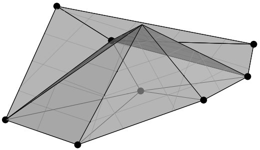

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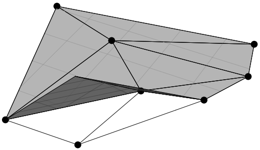

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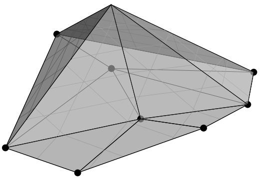

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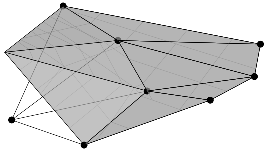

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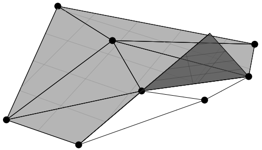

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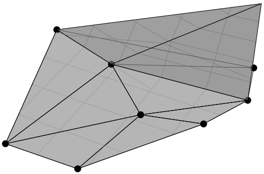

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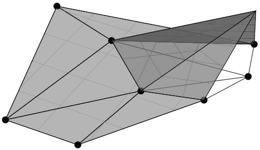

C^0 for triangulations: face rings

Face Ring of Δ

Δ a simplicial complex.

$$A_\Delta = \mathbb{R}[x_v | v \text{ a vertex of } \Delta]/I_\Delta,$$

where I_Δ is the ideal generated by monomials corresponding to non-faces.

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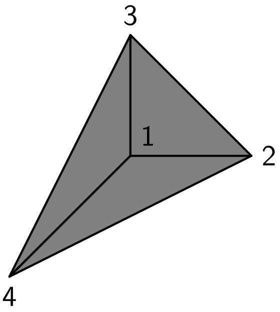

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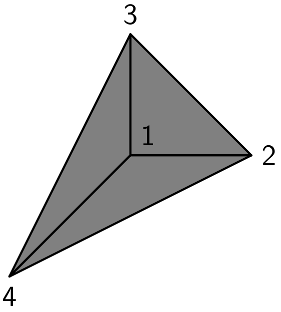

- Nonfaces are $\{1,2,3,4\}, \{2,3,4\}$
- $I_\Delta = \langle x_2x_3x_4 \rangle$
- $A_\Delta = \mathbb{R}[x_1, x_2, x_3, x_4]/I_\Delta$

C^0 for triangulations: the main structure theorem

C^0 for Simplicial Splines [Billera-Rose '92]

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Map is $T_v \to x_v$ (v not the cone vertex)
Consequences:

- $\dim C^0_d(\Delta) = \sum_{i=0}^{n} f_i \binom{d-1}{i}$ for $d > 0$, where $f_i = \#i$-faces of Δ.
- If Δ is homeomorphic to a disk, then $C^0(\widehat{\Delta})$ is free as a $S = \mathbb{R}[x_0, \ldots, x_n]$ module.
- If Δ is shellable, then degrees of free generators for $C^0(\widehat{\Delta})$ as S-module can be read off the h-vector of Δ.

Nonsimplicial Case

Nonfreeness for Polytopal Complexes [D. '12]

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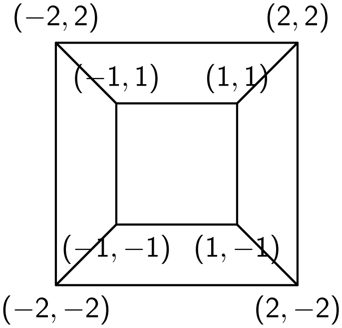

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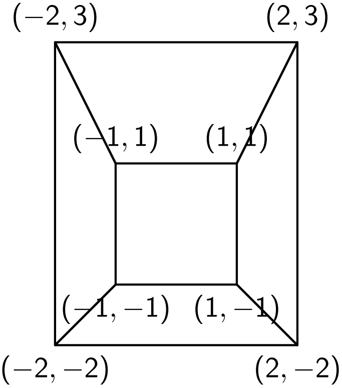

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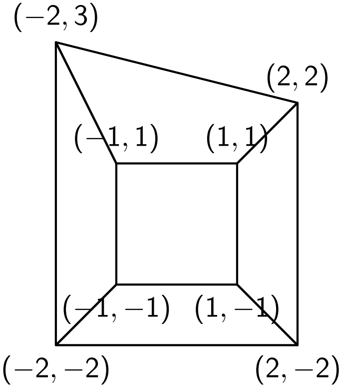

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Part II: Hyperplanes and Derivations

Hyperplane Arrangements

$\mathbb{K}$:	Field, characteristic zero
V :	$\mathbb{K}^\ell$
Hyperplane:	zero locus $V(\alpha)$ of affine linear form $\alpha = (\sum_{i=1}^{\ell} a_i x_i) + a_0$
Arrangement:	$\mathcal{A} = \cup_{i=1}^{k} H_i$, $H_i = V(\alpha_i)$.

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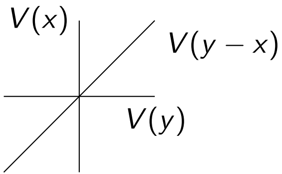
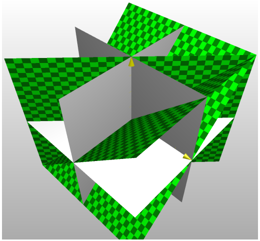
A_2 braid arrangement in $\mathbb{R}^2$:



Braid arrangement

Braid arrangement: $A_\ell = \bigcup_{0 \le i < j \le \ell} V(y_i - y_j) \subset \mathbb{K}^{\ell+1}$

Set $x_i = y_0 - y_i$: $A_\ell = \left(\bigcup_{i=1}^{\ell} V(x_i) \right) \cup \left(\bigcup_{1 \le i < j \le \ell} V(x_j - x_i) \right)$



A_3 braid arrangement in $\mathbb{R}^3$

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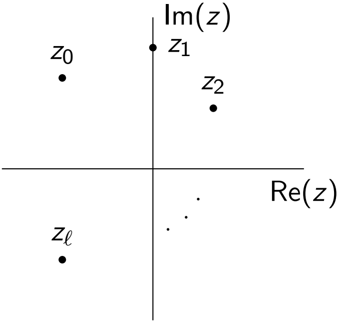

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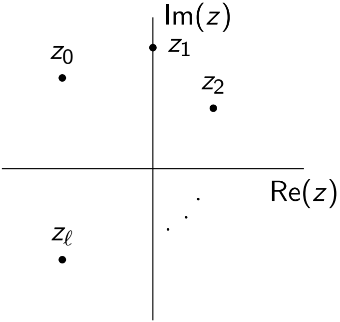

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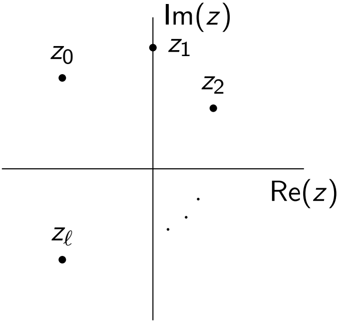

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Paths in $\mathbb{C}^{\ell+1} \setminus A_\ell \leftrightarrow$ non-colliding trajectories for $\ell + 1$ robots.

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Denote by $\pi(\mathcal{A}, t) := \sum_{i \geq 0} \mathrm{rk}(H^i(\mathbb{C}^\ell \setminus \mathcal{A}))t^i$ the *Poincare polynomial* of $\mathcal{A}$.

Module of derivations

- $V = \mathbb{K}^\ell, S = \mathrm{Sym}(V^*) \cong \mathbb{K}[x_1, \ldots, x_\ell]$

$$\mathrm{Der}_{\mathbb{K}}(S) := \left\{ \sum_{i=1}^{\ell} \theta_i \frac{\partial}{\partial x_i} : \theta_i \in S \text{ for } i = 1, \ldots, \ell \right\}$$

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- Given $f \in S$, $\theta = \sum \theta_i \frac{\partial}{\partial x_i} \in \mathrm{Der}_{\mathbb{K}}(S)$:
  - $f\theta = \sum (f\theta_i) \frac{\partial}{\partial x_i} \in \mathrm{Der}_{\mathbb{K}}(S)$ [$\mathrm{Der}_{\mathbb{K}}(S)$ is an S-module]
  - $\theta(f) = \sum \theta_i \frac{\partial f}{\partial x_i} \in S$

Module of $\mathcal{A}$-derivations

- $\mathcal{A} = \cup_{i=1}^{k} H_i$, $H_i = V(\alpha_i)$. Module of derivations of $\mathcal{A}$:

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- $D(\mathcal{A})$ is an S-module: $f \in S$, $\theta \in D(\mathcal{A})$, then $f\theta \in D(\mathcal{A})$
- $D(\mathcal{A})$ is a **free** S-module if there are $\theta_1, \ldots, \theta_\ell \in D(\mathcal{A})$ so that every $\theta \in D(\mathcal{A})$ can be written uniquely as $\theta = \sum_{i=1}^{\ell} f_i\theta_i$, where $f_i \in S$.
- $\mathcal{A}$ is **free** if $D(\mathcal{A})$ is a free S-module.

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Note: $\det \begin{bmatrix} x & y \\ x^2 & y^2 \end{bmatrix} = xy^2 - x^2y = xy(y-x)$ (Saito's Criterion!)

Consequence of Freeness

A seminal result of Terao relates freeness of $D(\mathcal{A})$ with the cohomology ring of $\mathbb{C}^\ell \setminus \mathcal{A}$.

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It is unknown precisely what characteristics make an arrangement free.

Lattice of an Arrangement

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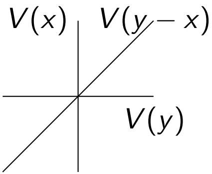
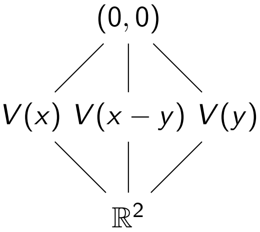


$\mathcal{A}$



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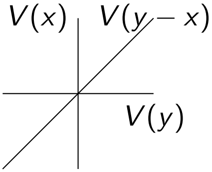

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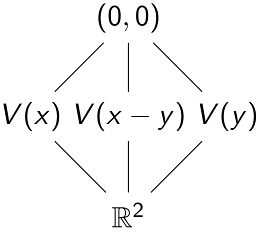

$L_{\mathcal{A}}$

A property of $\mathcal{A}$ is *combinatorial* if it only depends on $L_{\mathcal{A}}$.

Supersolvable Arrangements

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Open question: does freeness of $\mathcal{A}$ depend only on $L_{\mathcal{A}}$?

Part III: Connections

First Similarities

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- Correspondence extends to graphic *multi-arrangements* and study of *free multiplicities* on graphic arrangements
- Using the chain complex of Billera-Schenck-Stillman, get new obstructions to freeness of multi-braid arrangements [D-Francisco-Mermin-Schweig '16].

Hinting at deeper connections: Ziegler's Pair

Consider the one-parameter family of arrangements $\mathcal{A}_t$ whose hyperplanes are defined by the vanishing of the following forms (t is considered a parameter):

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Connection here hinges on *formality* of $\mathcal{A}_t$.

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3. Such homological obstructions complement known methods for proving freeness of arrangements via deletion-restriction.
4. Can deletion-restriction methods be found for splines?

THANK YOU!