# Extending Wilf's Conjecture

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Colloquium

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To eventually make all amounts, need $\gcd(a_1, \ldots, a_n) = 1$.

# Numerical Semigroups

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- *genus* is $g(S) = \#(\mathbb{N} \setminus S)$
- *Frobenius number* is $F(S) = \max(\mathbb{N} \setminus S)$
- *embedding dimension* is $e(S) = \min\{k : S = \langle a_1, \ldots, a_k \rangle\}$

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- Number theory (Frobenius problems)

# Two coins

## Sylvester 1884

If $\gcd(a, b) = 1$ and $S = \langle a, b \rangle$, then

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$S$ is *symmetric*: $x \to F - x$ gives bijection between holes and non-holes

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3. $F(S) \geq \sqrt{3abc} - a - b - c$ [Davison '94]
4. On average, $F(S) \approx \frac{8}{\pi}\sqrt{abc} - a - b - c$ [Ustinov '09]

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## Wilf '78 (American Mathematical Monthly)

1. (Wilf's conjecture) If $S = \langle a_1, \ldots, a_k \rangle$, is it true that

$$\frac{\#N(S)}{\#C(S)} = \frac{F + 1 - g(S)}{F + 1} \geq \frac{1}{k}?$$

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$$\frac{\#N(S)}{\#C(S)} = \frac{F + 1 - g(S)}{F + 1} \geq \frac{1}{k}?$$
2. Let $N_g = $ number of semigroups with genus $g$. What is the growth rate of $N_g$?

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Wilf's conjecture is satisfied with equality!

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All $S$ with embedding dimension three satisfy Wilf's conjecture [Fröberg, Gottlieb, and Häggkvist '87].

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4. True if $\min(S \setminus \{0\}) \leq 18$ [Bruns, García-Sánchez, O'Neill, Wilburne '19]

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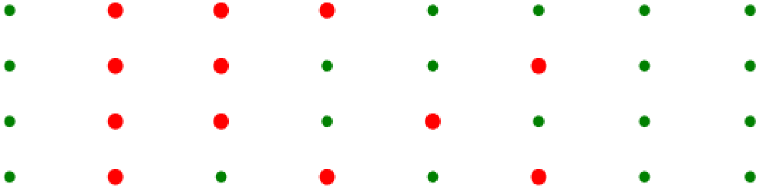

# Fibonacci-like growth

| $g$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
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- (2) and (3) proved by Zhai (2013). (1) is still open
- $N_g > N_{g-1}$ still open (holds for $g \gg 0$)

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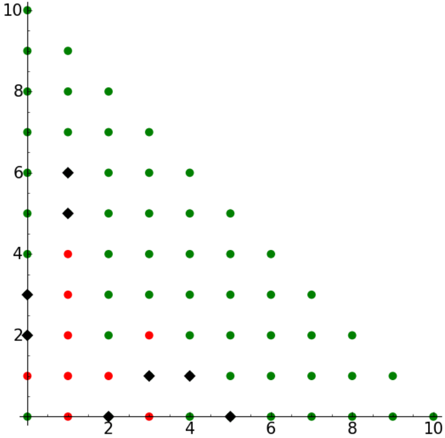

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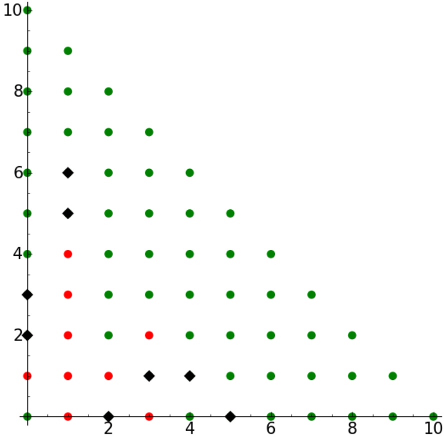

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**Generalized Wilf Conjecture [Cisto,-, Failla, Flores, Peterson, Utano '19]**

If $S \subset \mathbb{N}^d$ is a GNS with $S = \langle \mathbf{a}_1, \ldots, \mathbf{a}_k \rangle$ then

$$\frac{\#N(S)}{\#C(S)} \ge \frac{d}{k}$$

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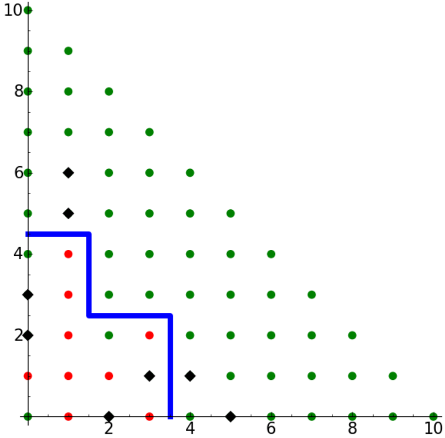

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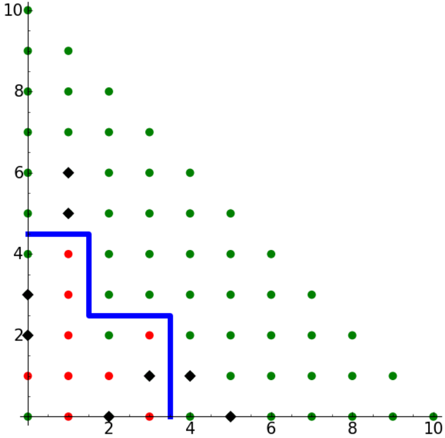


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$L =$ jagged line

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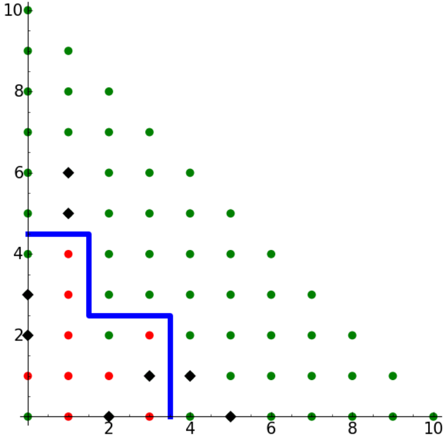


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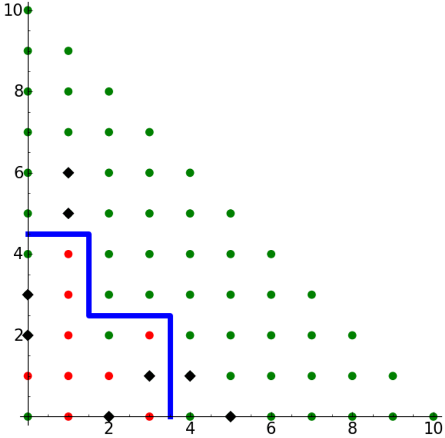


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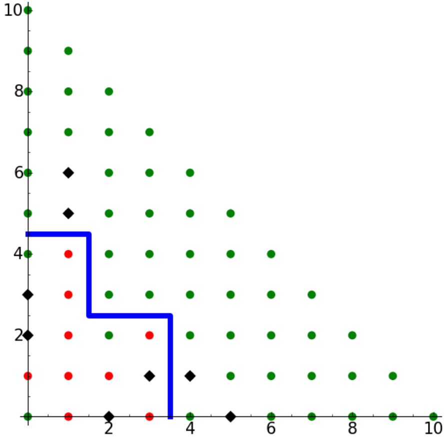


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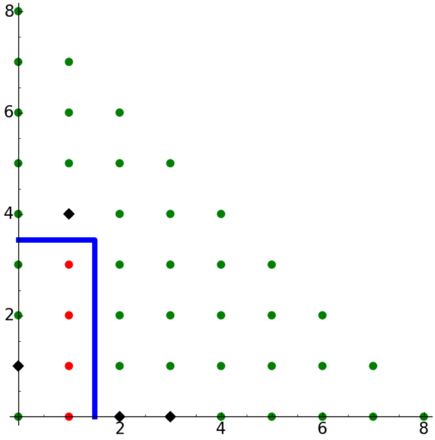

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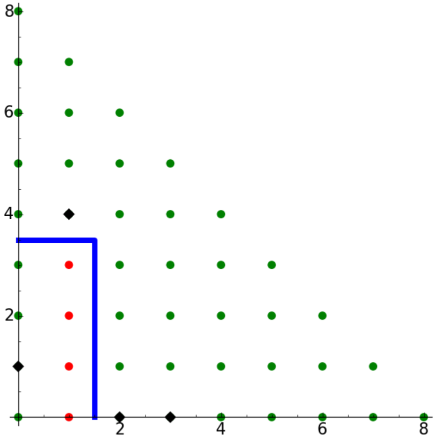


$S = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \rangle$ columns of

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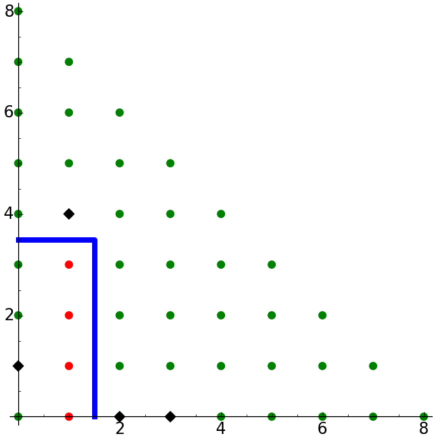


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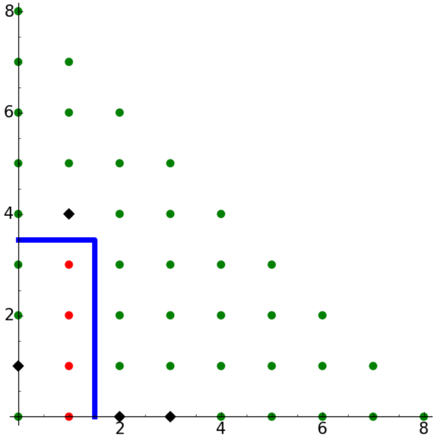


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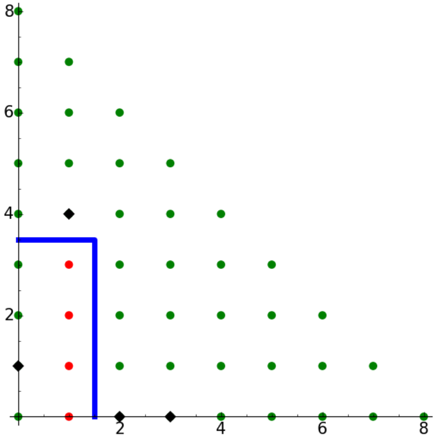


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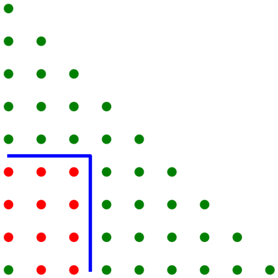

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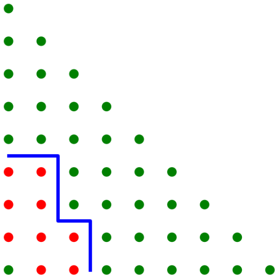

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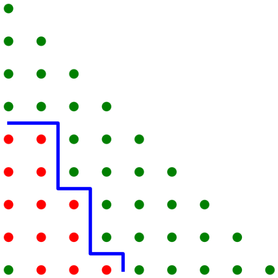

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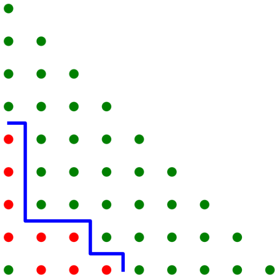

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- Another extension of Wilf's conjecture is given which depends on a monomial order and does not incorporate dimension

# An affine semigroup and its holes

$$S = \{A\mathbf{x} : \mathbf{x} \in \mathbb{N}^k\}$$

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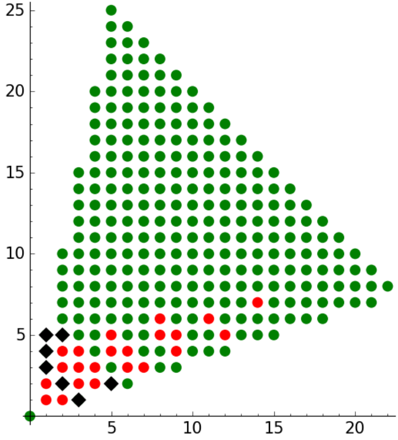

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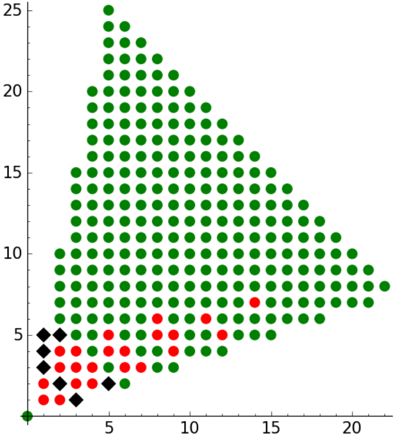

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1. Frobenius numbers for affine semigroups – still widely open [Aliev, De Loera, Louveaux '16]
2. Is there a version of Wilf's conjecture for arbitrary affine semigroups (with infinitely many holes)?
3. Fix a pointed cone $\mathcal{C}$. Let $N_{\mathcal{C},g}$ be the number of affine semigroups with $g$ holes and conical hull $\mathcal{C}$. What is the rate of growth of $N_{\mathcal{C},g}$ with respect to $g$? (first addressed for $\mathcal{C} = \mathbb{N}^d$ in [Failla, Peterson, and Utano '16])