Commutative Algebra and Approximation Theory

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Colloquium

Part I: Background and Central Questions

Piecewise Polynomials

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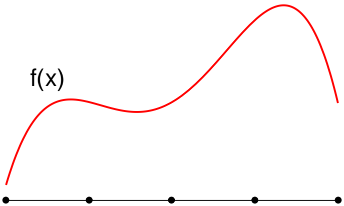

Graph of a function

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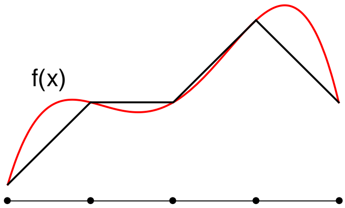

Trapezoid Rule

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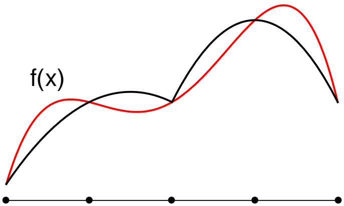

Simpson's Rule

Univariate Splines

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- Subdivide $\Delta = [a, b]$ into subintervals:
  $\Delta = [a_0, a_1] \cup [a_1, a_2] \cup \cdots \cup [a_{n-1}, a_n]$
- Find a basis for the vector space $C^r_d(\Delta)$ of C^r piecewise polynomial functions on Δ with degree at most d (e.g. B-splines)
- Find best approximation to $f(x)$ in $C^r_d(\Delta)$

Two Subintervals

$\Delta = [a_0, a_1] \cup [a_1, a_2]$ (assume WLOG $a_1 = 0$)

$$\begin{aligned}(f_1, f_2) \in C^r_d(\Delta) &\iff f_1^{(i)}(0) = f_2^{(i)}(0) \text{ for } 0 \leq i \leq r\\ &\iff x^{r+1} | (f_2 - f_1)\\ &\iff (f_2 - f_1) \in \langle x^{r+1} \rangle\end{aligned}$$

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Even more explicitly:

- $f_1(x) = b_0 + b_1 x + \cdots + b_d x^d$
- $f_2(x) = c_0 + c_1 x + \cdots + c_d x^d$
- $(f_0, f_1) \in C^r_d(\Delta) \iff b_0 = c_0, \ldots, b_r = c_r.$

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- $(f_0, f_1) \in C^r_d(\Delta) \iff b_0 = c_0, \ldots, b_r = c_r.$

$$\dim C^r_d(\Delta) = \begin{cases} d+1 & \text{if } d \leq r \\ (d+1)+(d-r) & \text{if } d > r \end{cases}$$

Note: $\dim C^r_d(\Delta)$ is polynomial in d for $d > r$.

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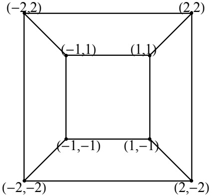

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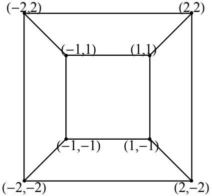

A polytopal complex $\mathcal{Q}$

(Algebraic) Spline Criterion:

- If $\tau \in \Delta_{n-1}$, $l_\tau =$ affine form vanishing on affine span of τ
- Collection $\{F_\sigma\}_{\sigma \in \Delta_n}$ glue to $F \in C^r(\Delta) \iff$ for every pair of adjacent facets $\sigma_1, \sigma_2 \in \Delta_n$ with $\sigma_1 \cap \sigma_2 = \tau \in \Delta_{n-1}$, $l_\tau^{r+1} | (F_{\sigma_1} - F_{\sigma_2})$

The dimension question

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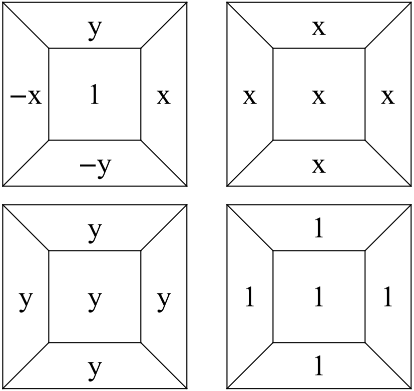
$\dim_{\mathbb{R}} C^0_1(\mathcal{Q}) = 4$



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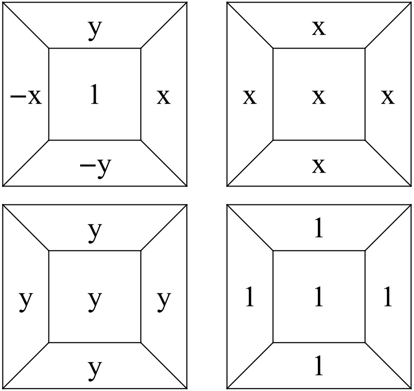


Two central problems in approximation theory:

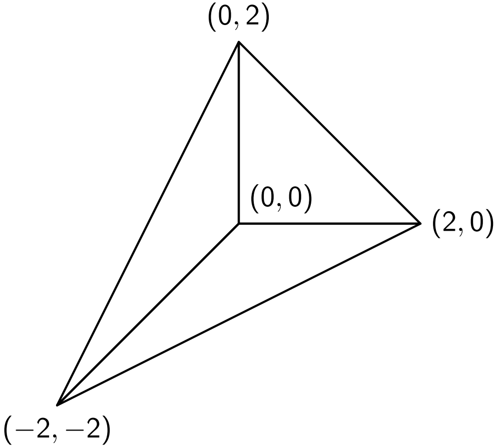
1. Determine $\dim C^r_d(\Delta)$
2. Construct a 'local' basis of $C^r_d(\Delta)$, if possible

Who Cares?

1. Computation of dim $C^r_d(\Delta)$ for higher dimensions initiated by [Strang '75] in connection with finite element method
2. Data fitting in approximation theory
3. Computer Aided Geometric Design (CAGD) - building surfaces by splines [Farin '97]
4. Toric Geometry: Equivariant Chow cohomology rings of toric varieties are rings of continuous splines on the fan (under appropriate conditions) [Payne '06], more generally GKM theory

Part II: How is commutative algebra useful?

Continuous Splines in Two Dimensions



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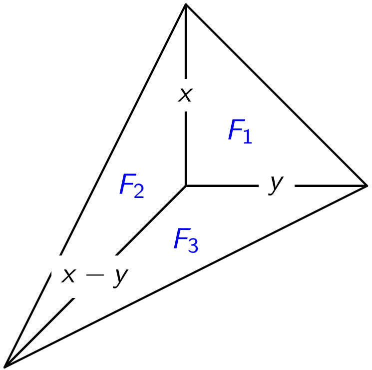

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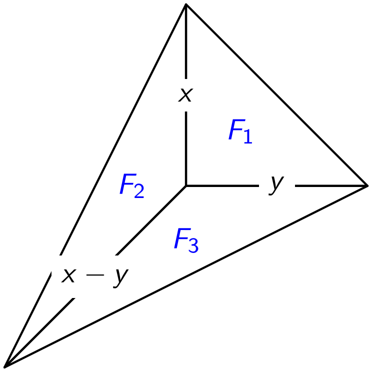
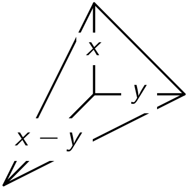


$(F_1, F_2, F_3) \in C^0(\Delta) \iff$ $\exists f_1, f_2, f_3$ so that

$$\begin{aligned} F_1 - F_2 &= f_1 x \\ F_2 - F_3 &= f_2(x-y) \\ F_3 - F_1 &= f_3 y \end{aligned}$$

Freeness



Three splines in $C^0(\Delta)$:

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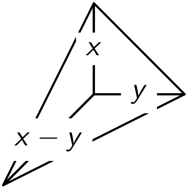

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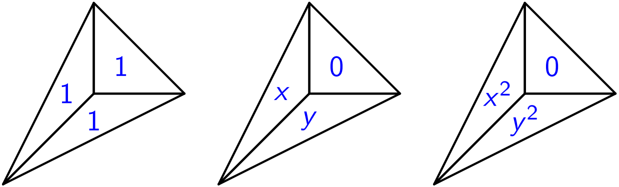

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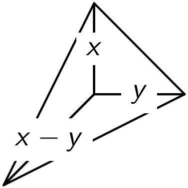

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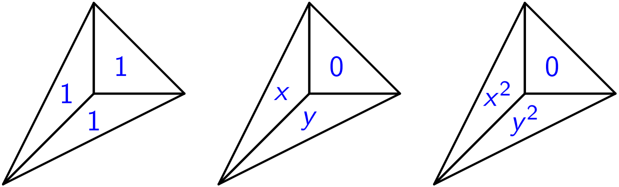

- In fact, every spline $F \in C^0(\Delta)$ can be written uniquely as a polynomial combination of these three splines.

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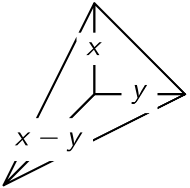

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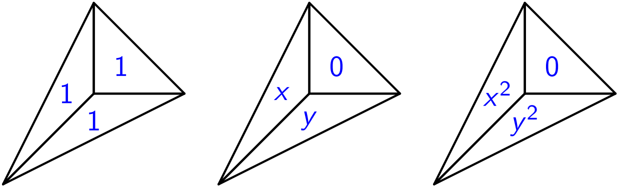

- In fact, every spline $F \in C^0(\Delta)$ can be written uniquely as a polynomial combination of these three splines.
- We say $C^0(\Delta)$ is a **free** $\mathbb{R}[x, y]$-module, generated in **degrees** $0, 1, 2$

Freeness and Dimension Computation

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- $C^0_d(\Delta) \cong \mathbb{R}[x,y]_{\le d}(1,1,1) \oplus \mathbb{R}[x,y]_{\le d-1}(0,x,y) \oplus \mathbb{R}[x,y]_{\le d-2}(0,x^2,y^2)$.
- $\dim C^0_d(\Delta) = \binom{d+2}{2} + \binom{d+1}{2} + \binom{d}{2}$
  $= \frac{3}{2}d^2 + \frac{3}{2}d + 1$ for $d \ge 1$

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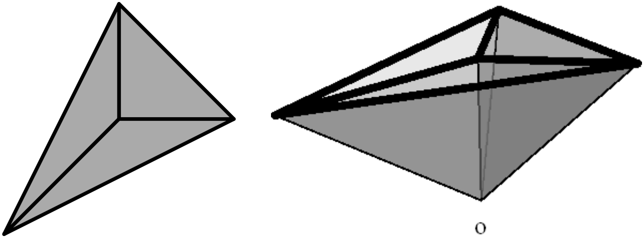
- $\dim C^0_d(\Delta) = \binom{d+2}{2} + \binom{d+1}{2} + \binom{d}{2}$ $= \frac{3}{2}d^2 + \frac{3}{2}d + 1$ for $d \ge 1$

In general, employ a *coning* construction $\Delta \to \widehat{\Delta}$ to homogenize and consider $\dim C^r(\widehat{\Delta})_d$.

Coning Construction

- $\widehat{\Delta} \subset \mathbb{R}^{n+1}$ denotes the cone over $\Delta \subset \mathbb{R}^n$.



Δ $\widehat{\Delta}$

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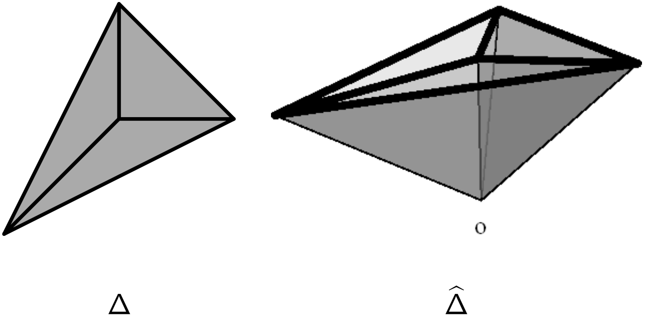

- $C^r(\widehat{\Delta})$ is always a **graded** module over $\mathbb{R}[x_0, \ldots, x_n]$
- $C^r_d(\Delta) \cong C^r(\widehat{\Delta})_d$ [Billera-Rose '91]

Hilbert series and polynomial

From commutative algebra

- $\dim C^r_d(\Delta) = \dim C^r(\widehat{\Delta})_d$ is called the *Hilbert function* of $C^r(\widehat{\Delta})$; it is a polynomial in d for $d \gg 0$
- This is called the *Hilbert polynomial* of $C^r(\widehat{\Delta})$, denoted $HP(C^r(\widehat{\Delta}), d)$

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- The *Hilbert series* is the formal sum $HS(C^r(\widehat{\Delta}), t) = \sum_{d=0}^{\infty} \dim C^r_d(\Delta) t^d$; it has the form

$$HS(C^r(\widehat{\Delta}), t) = \frac{h(t)}{(1-t)^{d+1}}, \text{ where } h(t) \in \mathbb{Z}[t].$$

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Main questions:

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Main questions:

- Determine $HS(C^r(\widehat{\Delta}), t)$. (too hard!)
- What is a formula for $HP(C^r(\widehat{\Delta}), d)$?
- How large must d be so that $\dim C^r_d(\Delta) = HP(C^r(\widehat{\Delta}), d)$?

Part III: The planar dimension formulas

Planar simplicial splines of large degree

Planar simplicial dimension [Alfeld-Schumaker '90]

If $\Delta \subset \mathbb{R}^2$ is a simply connected triangulation and $d \geq 3r+1$,

$$\dim C^r_d(\Delta) = f_2\binom{d+2}{2} - f_1^0\left(\binom{d+2}{2} - \binom{d-r+1}{2}\right) + \sigma,$$

- $f_i(f_i^0)$ is the number of i-faces (interior i-faces).
- σ = constant obtained as a sum of contributions from each interior vertex.

Planar non-simplicial splines of large degree

Planar non-simplicial dimension [McDonald-Schenck '09]

If $\Delta \subset \mathbb{R}^2$ is a simply connected polytopal complex and $d \gg 0$,

$$\dim C^r_d(\Delta) = f_2\binom{d+2}{2} - f_1^0\left(\binom{d+2}{2} - \binom{d-r+1}{2}\right) + \sigma + \sigma',$$

- $f_i(f_i^0)$ is the number of i-faces (interior i-faces).
- $\sigma =$ sum of constant contributions from interior vertices
- $\sigma' =$ sum of constant contributions from 'missing' vertices

Dimension computation via homology

Use Billera-Schenck-Stillman chain complex to derive Hilbert polynomial:

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$$\mathcal{R}/\mathcal{J}: \quad 0 \longrightarrow \bigoplus_{\sigma\in\Delta_2} S \xrightarrow{\partial_2} \bigoplus_{\tau\in\Delta_1^0} \frac{S}{J(\tau)} \xrightarrow{\partial_1} \bigoplus_{v\in\Delta_0^0} \frac{S}{J(v)} \longrightarrow 0,$$

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∂_2, ∂_1 : cellular differentials of Δ relative to boundary

- $\ker(\partial_2) = C^r(\Delta)$
- Via coning/homogenizing, all modules can be made graded
- Euler characteristic:

$$\dim C^r_d(\Delta) = |\Delta_2| \cdot \dim S_d - \sum_{\tau\in\Delta_1^\circ} \dim \left(\frac{S}{J(\tau)}\right)_d + \sum_{v\in\Delta_0^\circ} \dim \left(\frac{S}{J(v)}\right)_d + \dim H_1(\mathcal{R}/\mathcal{J})_d - \dim H_2(\mathcal{R}/\mathcal{J})_d$$

Dimension computation via homology, continued

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- $\dim(S/J(v))_d$ computed by [Schumaker], [Stiller '83], [Schenck '97], . . .

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- $\dim(S/J(v))_d$ computed by [Schumaker], [Stiller '83], [Schenck '97], ...
- $H_2(\mathcal{R}/\mathcal{J}) = 0$
- $HP(H_1(\mathcal{R}/\mathcal{J}), d)$ determined via localization - either vanishes (simplicial, generic polytopal cases) or is constant [McDonald-Schenck '09]

Dimension computation via homology, continued

$$\dim C^r_d(\Delta) = |\Delta_2| \cdot \dim S_d - \sum_{\tau \in \Delta_1^\circ} \dim \left(\frac{S}{J(\tau)}\right)_d + \sum_{v \in \Delta_0^\circ} \dim \left(\frac{S}{J(v)}\right)_d + \dim H_1(\mathcal{R}/\mathcal{J})_d - \dim H_2(\mathcal{R}/\mathcal{J})_d$$

- $\dim(S/J(v))_d$ computed by [Schumaker], [Stiller '83], [Schenck '97], ...
- $H_2(\mathcal{R}/\mathcal{J}) = 0$
- $HP(H_1(\mathcal{R}/\mathcal{J}), d)$ determined via localization - either vanishes (simplicial, generic polytopal cases) or is constant [McDonald-Schenck '09]

Remark: For $\Delta \subset \mathbb{R}^3$, computing $\dim(S/J(v))_d$ for $v \in \Delta_0^\circ$ translates to computing dimension of fat point schemes in $\mathbb{P}^2$ (**much** harder than planar setting - see for instance the Segre-Harbourne-Gimigliano-Hirschowitz (SHGH) Conjecture).

Agreement for non-simplicial splines

How large must d be in order for $HP(C^r(\widehat{\Delta}), d) = \dim C^r_d(\Delta)$?

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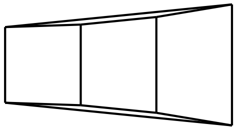

$\dim C^0_d(\Delta) = \frac{5}{2}d^2 - \frac{1}{2}d + 1$ for $d \geq 2$

(By Theorem must have agreement for $d \geq 6$)

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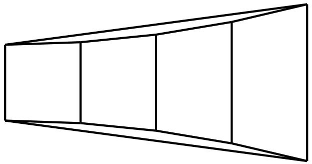

$\dim C^0_d(\Delta) = \frac{6}{2}d^2 - \frac{4}{2}d + 1$ for $d \geq 3$

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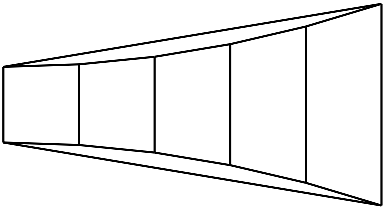

$\dim C^0_d(\widehat{\Delta}) = \frac{7}{2}d^2 - \frac{7}{2}d + 1$ for $d \geq 4$

(By Theorem must have agreement for $d \geq 10$)

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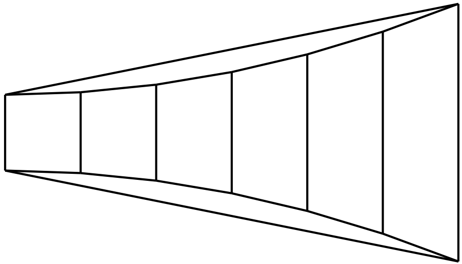
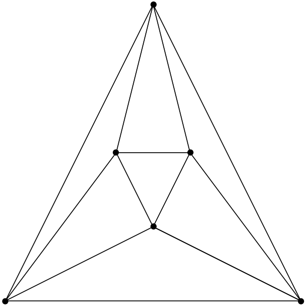


$\dim C^0_d(\Delta) = \frac{8}{2}d^2 - \frac{10}{2}d + 1$ for $d \geq 5$

(By Theorem must have agreement for $d \geq 12$)

Low Degree: Morgan-Scot triangulation



$\dim C^1_2(\mathcal{T}) = 7$

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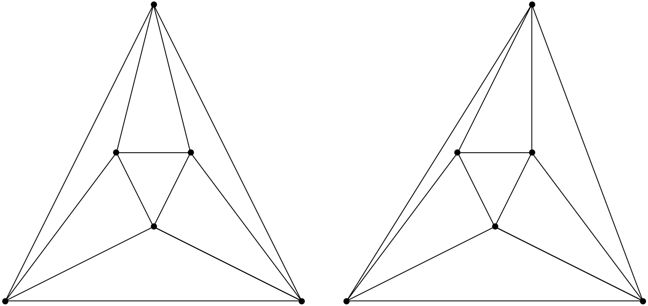

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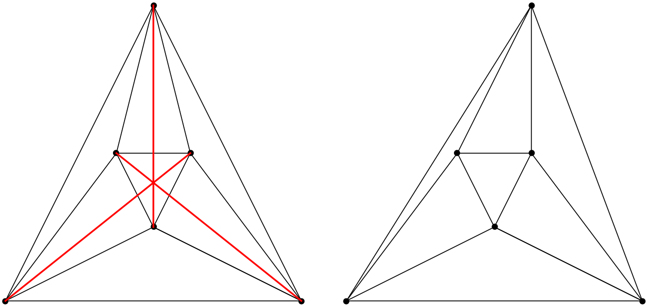

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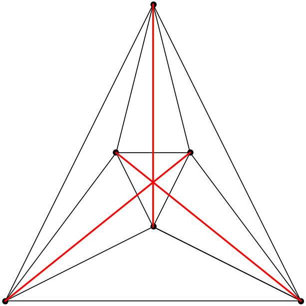

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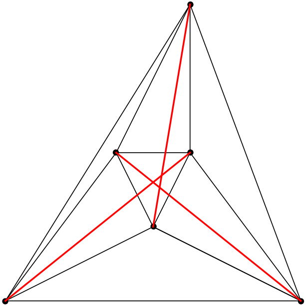

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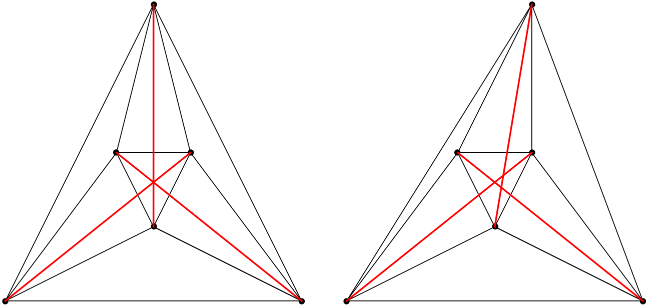

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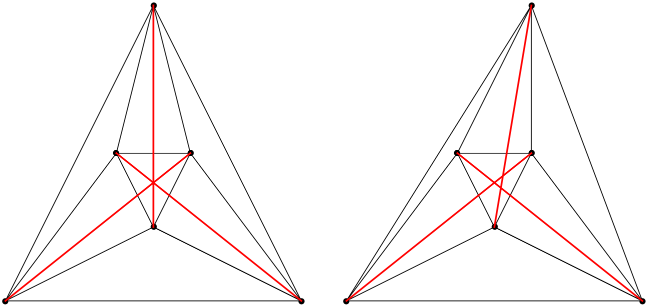

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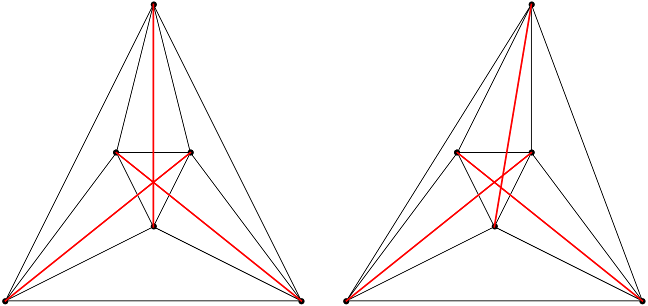

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Part IV: Freeness

Freeness

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- (Δ a planar triangulation) $C^1_d(\Delta) =$ expected dimension for $d \geq 2$ if and only if $C^1(\widehat{\Delta})$ is free.

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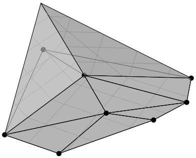

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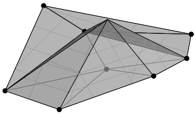

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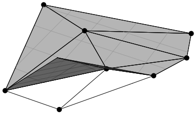

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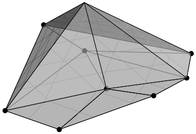

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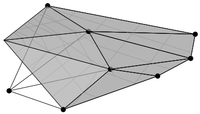

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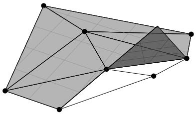

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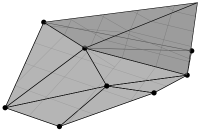

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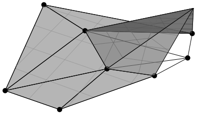

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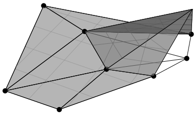

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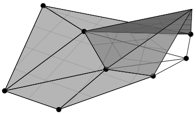

- dim $C^0_1(\Delta)$ = number of vertices of Δ
- $C^0(\Delta)$ is generated as an algebra by tent functions [Billera-Rose ‘92]

Face rings of simplicial complexes

Face Ring of Δ

Δ a simplicial complex.

$$A_\Delta = \mathbb{R}[x_v | v \text{ a vertex of } \Delta]/I_\Delta,$$

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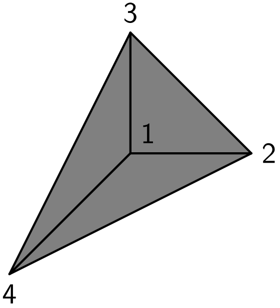

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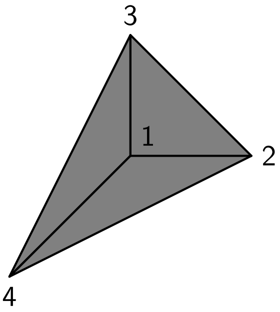

- Nonfaces are $\{1, 2, 3, 4\}, \{2, 3, 4\}$
- $I_\Delta = \langle x_2 x_3 x_4 \rangle$
- $A_\Delta = \mathbb{R}[x_1, x_2, x_3, x_4]/I_\Delta$

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Consequences:

- $C^0(\widehat{\Delta})$ is entirely combinatorial!
- $\dim C^0_d(\Delta) = \sum_{i=0}^{n} f_i \binom{d-1}{i}$ for $d > 0$, where $f_i = \#i$-faces of Δ.
- If Δ is homeomorphic to a disk, then $C^0(\widehat{\Delta})$ is free as a $S = \mathbb{R}[x_0, \ldots, x_n]$ module.
- If Δ is shellable, then degrees of free generators for $C^0(\widehat{\Delta})$ as S-module can be read off the h-vector of Δ.

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Nonfreeness for Polytopal Complexes [D. '12]

$C^0(\widehat{\Delta})$ need not be free if Δ has nonsimplicial faces [D. '12].

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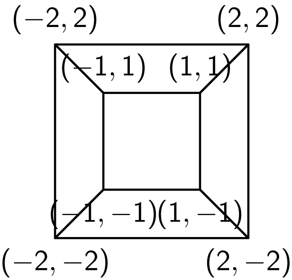

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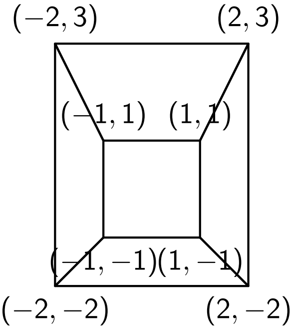

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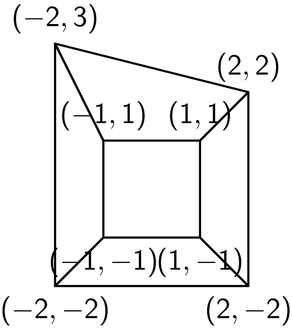

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A partition of a domain D is called a *cross-cut partition* if the union of its two-cells are the complement of a line arrangement.

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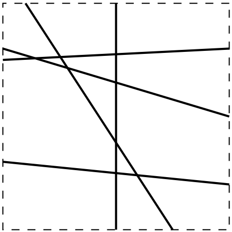

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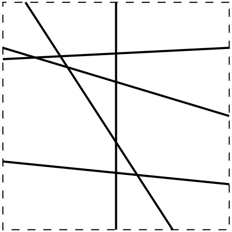

- Basis for $C^r_d(\Delta)$ and dim $C^r_d(\Delta)$ [Chui-Wang '83]
- $C^r(\widehat{\Delta})$ is free for any r [Schenck '97]

Cautionary Tale II: Ziegler's Pair

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$\mathcal{A}_t$ = union of hyperplanes defined by the vanishing of the forms (t is considered a parameter):

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$\mathcal{A}_t$ has six triple lines (where three planes intersect) for most choices of t; these lie on a non-degenerate conic if $t = 0$.

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$\mathcal{A}_t$ has six triple lines (where three planes intersect) for most choices of t; these lie on a non-degenerate conic if $t = 0$.

- Let Δ_t be the polytopal complex formed by closures of connected components of $[-1,1] \times [-1,1] \times [-1,1] \setminus \mathcal{A}_t$. (there are 62 polytopes)

Cautionary Tale II: Ziegler's Pair

Cross-cut partitions fail to be free in $\mathbb{R}^3$!

$\mathcal{A}_t$ = union of hyperplanes defined by the vanishing of the forms (t is considered a parameter):

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y	2x+3y+z	2x+3y+4z
z	(1+t)x+(3+t)z	(1+t)x+(2+t)y+(3+t)z

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- Let Δ_t be the polytopal complex formed by closures of connected components of $[-1,1] \times [-1,1] \times [-1,1] \setminus \mathcal{A}_t$. (there are 62 polytopes)
- $C^0(\Delta_t)$ is free if and only if $t \neq 0$!

Part V: Open Questions

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- More generally (planar polytopal complexes): Compute $\dim C^r_d(\Delta)$ for $r+1 \leq d \leq (2F-1)(r+1)$ (F maximum number of edges in a two-cell)
- If $\Delta \subset \mathbb{R}^3$, $\dim C^r_d(\Delta)$ is not known for $d \gg 0$ except for $r = 1$, $d \geq 8$ on generic triangulations [Alfeld-Schumaker-Whitely '93]. (connects to fat point schemes in $\mathbb{P}^2$)

Open Questions

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- Characterize freeness $C^r(\Delta)$. Start with C^0 splines on cross-cut partitions Δ in $\mathbb{R}^3$.
- Compute $\dim C^r_d(\Delta)$ for semi-algebraic splines on planar partitions for $d \gg 0$

THANK YOU!

Part V: Semi-algebraic Splines

Curved Partitions

More general problem: Compute $\dim C^r_d(\Delta)$ where Δ is a partition whose arcs consist of irreducible algebraic curves.

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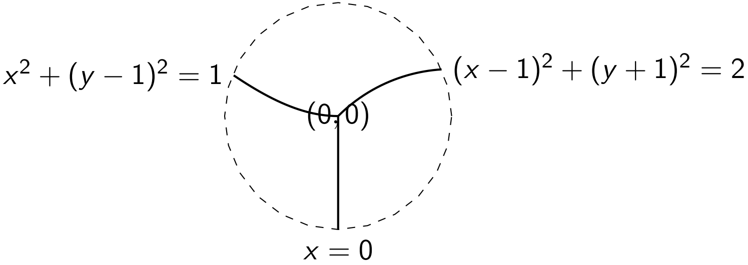

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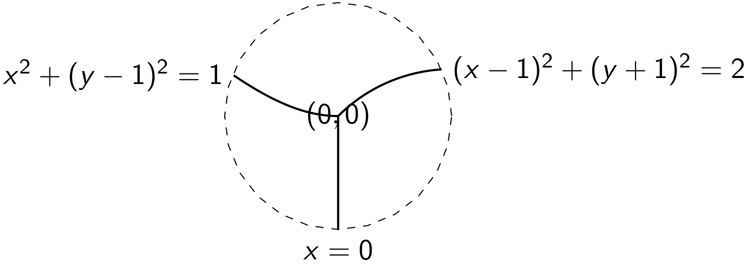

Call functions in $C^r(\Delta)$ *semi-algebraic splines* since they are defined over regions given by polynomial inequalities, or semi-algebraic sets.

Semi-algebraic Splines

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- Studied using sheaf-theoretic techniques [Stiller '83]
- Recent work suggests semi-algebraic splines may be increasingly useful in finite element method [Davydov-Kostin-Saeed '16]

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- Focus on $\Delta \subset \mathbb{R}^2$ with single interior vertex at $(0, 0)$.
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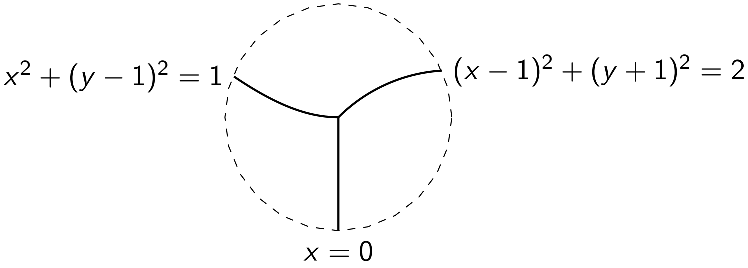

Δ

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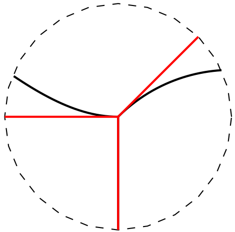

Tangent Lines

Linearizing

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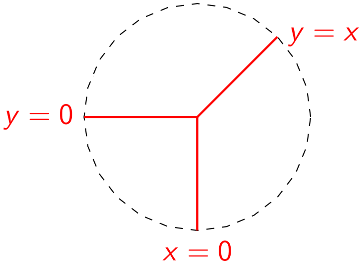

Δ_L

Linearizing the local case

Theorem: Linearizing dim $C^r_d(\Delta)$ [D.-Sottile-Sun '16]

Let Δ consist of n irreducible curves of degree $d_1, \ldots, d_n$ meeting at $(0,0)$ with distinct tangents and no common zero in $\mathbb{P}^2(\mathbb{C})$ other than $(0,0)$. Then, for $d \gg 0$,

$$\dim C^r_d(\Delta) = \dim C^r_d(\Delta_L) + \sum_{i=1}^{n} \left(\binom{d+2-d_i(r+1)}{2} - \binom{d-r-1}{2} \right)$$

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- Bounds on d for when equality holds are also considered, using regularity