Piecewise Polynomials and Algebraic Geometry

Michael DiPasquale
University of Idaho
Colloquium

Piecewise Polynomials

Spline

A piecewise polynomial function, continuously differentiable to some order.

Univariate Splines

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- Subdivide $\Delta = [a, b]$ into subintervals:
  $\Delta = [a_0, a_1] \cup [a_1, a_2] \cup \cdots \cup [a_{n-1}, a_n]$
- Find a basis for the vector space $C^r_d(\Delta)$ of C^r piecewise polynomial functions on Δ with degree at most d (B-splines!)
- Find best approximation to $f(x)$ in $C^r_d(\Delta)$

Two Subintervals

$\Delta = [a_0, a_1] \cup [a_1, a_2]$ (assume WLOG $a_1 = 0$)

$$\begin{aligned} (f_1, f_2) \in C^r_d(\Delta) &\iff f_1^{(i)}(0) = f_2^{(i)}(0) \text{ for } 0 \le i \le r \\ &\iff x^{r+1} | (f_2 - f_1) \\ &\iff (f_2 - f_1) \in \langle x^{r+1} \rangle \end{aligned}$$

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Even more explicitly:

- $f_1(x) = b_0 + b_1 x + \cdots + b_d x^d$
- $f_2(x) = c_0 + c_1 x + \cdots + c_d x^d$
- $(f_0, f_1) \in C^r_d(\Delta) \iff b_0 = c_0, \ldots, b_r = c_r.$

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$$\dim C^r_d(\Delta) = \begin{cases} d+1 & \text{if } d \le r \\ (d+1) + (d-r) & \text{if } d > r \end{cases}$$

Note: $\dim C^r_d(\Delta)$ is polynomial in d for $d > r$.

Univariate Dimension Formula

Suppose I is a subdivision of an interval with v^0 interior vertices and e edges. Then

$$\dim C^r_d(I) = \begin{cases} d+1 & d < r+1 \\ e(d+1) - v^0(r+1) & d \geq r+1 \end{cases}$$

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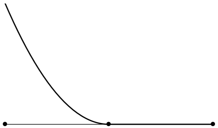

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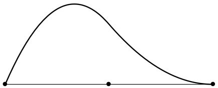

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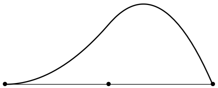

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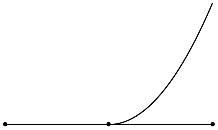

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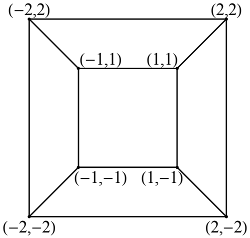

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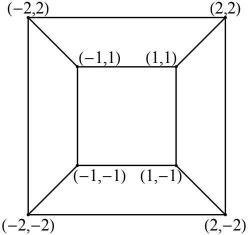

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(Algebraic) Spline Criterion:

- For $\tau \in \Delta_{n-1}$, l_τ = affine form vanishing on affine span of τ
- Collection $\{f_\sigma\}_{\sigma \in \Delta_n}$ glue to $F \in C^r(\Delta) \iff$ for every pair of adjacent facets $\sigma_1, \sigma_2 \in \Delta_n$ with $\sigma_1 \cap \sigma_2 = \tau \in \Delta_{n-1}$, $l_\tau^{r+1} | (f_{\sigma_1} - f_{\sigma_2})$

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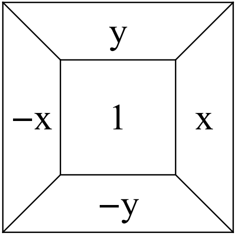

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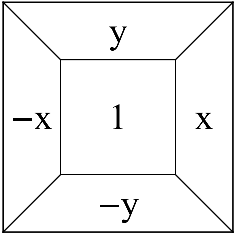

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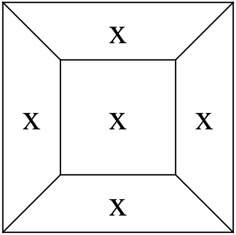

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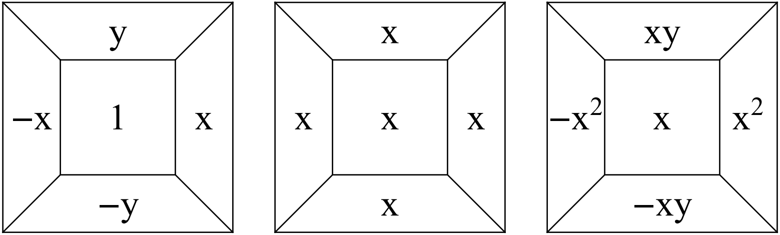


$F \in C^0_1(\mathcal{Q})$ $\quad x \in C^0_1(\mathcal{Q})$ $\quad xF \in C^0_2(\mathcal{Q})$

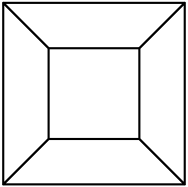
Who Cares?

1. Computation of dim $C^r_d(\Delta)$ for higher dimensions initiated by [Strang '73] in connection with finite element method
2. Data fitting in approximation theory
3. [Farin '97] Computer Aided Geometric Design (CAGD) - building surfaces by splines.
4. [Payne '06] Toric Geometry - Equivariant cohomology rings of toric varieties are rings of continuous splines on the fan (under appropriate conditions).

Part I: Continuous Splines and (some) C^1 Splines

Prelude: Coning Construction

- $\widehat{\Delta} \subset \mathbb{R}^{n+1}$ denotes the cone over $\Delta \subset \mathbb{R}^n$.



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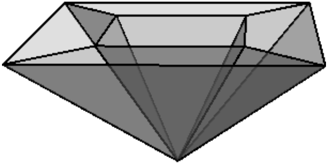

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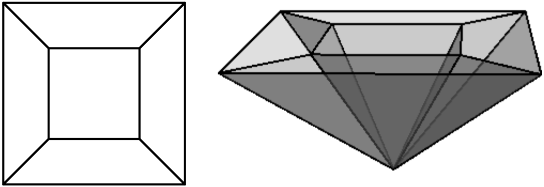
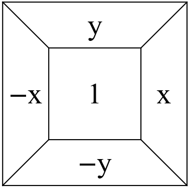


$\mathcal{Q}$ $\qquad\qquad$ $\widehat{\mathcal{Q}}$

- $C^r(\widehat{\Delta})$ is a **graded** module over $S = \mathbb{R}[x_0, \ldots, x_n]$ (every spline can be written as a sum of *homogeneous* splines)
- $F(x_0, \ldots, x_n) \in C^r(\widehat{\Delta}) \to F(1, x_1, \ldots, x_n) \in C^r(\Delta)$
- In fact $C^r_d(\Delta)$ (splines of degree at most d)$\cong C^r(\widehat{\Delta})_d$ (splines of degree exactly d) [Billera-Rose '91].

Coning example



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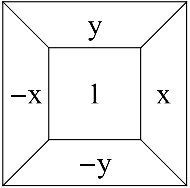

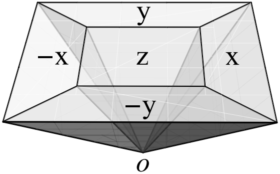

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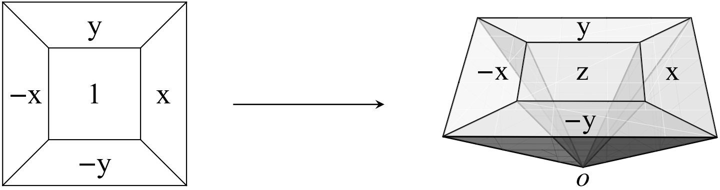

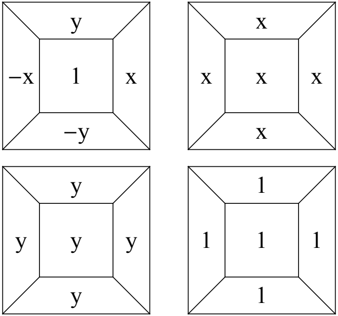

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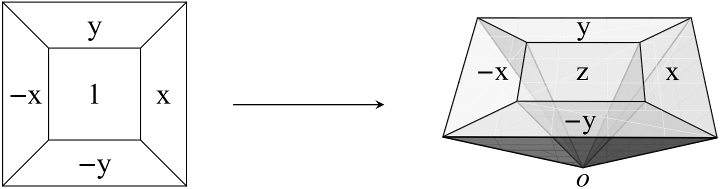

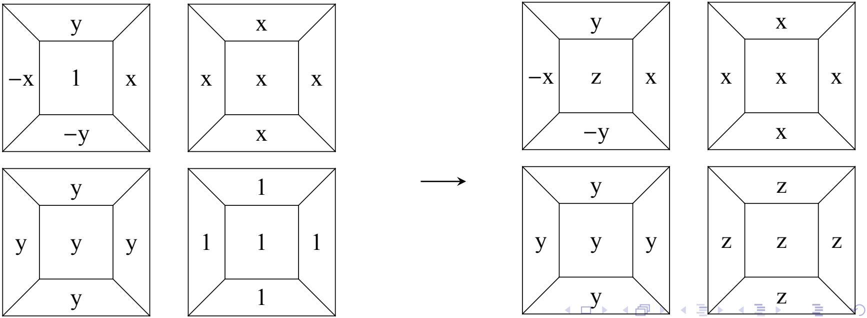

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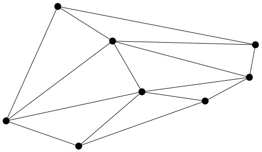

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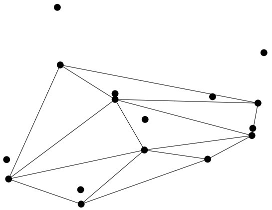

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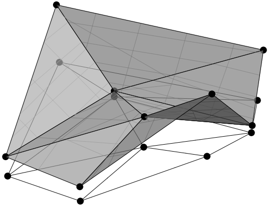

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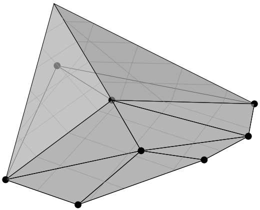

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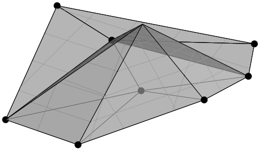

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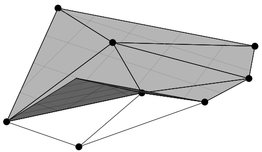

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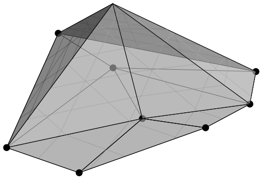

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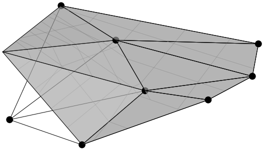

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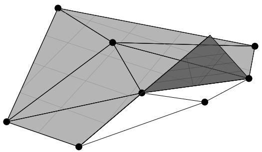

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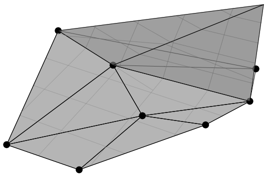

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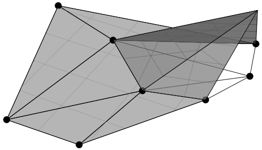

Face Rings of Simplicial Complexes

Face Ring of Δ

Δ a simplicial complex.

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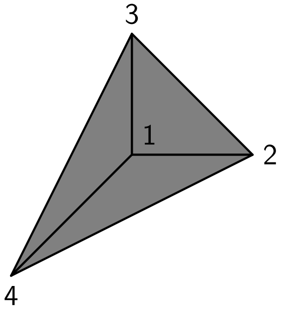

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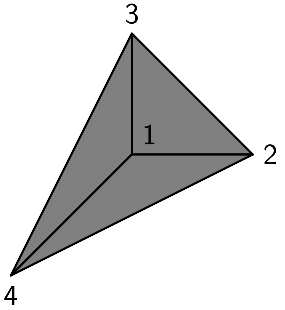

- Nonfaces are $\{1,2,3,4\}, \{2,3,4\}$
- $I_\Delta = \langle x_2 x_3 x_4 \rangle$
- $A_\Delta = \mathbb{R}[x_1, x_2, x_3, x_4]/I_\Delta$

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Consequences:

- $\dim C^0_d(\Delta) = \sum_{i=0}^{n} f_i \binom{d-1}{i}$ for $d > 0$, where $f_i = \#i$-faces of Δ.
- If Δ is homeomorphic to a disk, then $C^0(\widehat{\Delta})$ is free as a $S = \mathbb{R}[x_0, \ldots, x_n]$ module.
- If Δ is shellable, then degrees of free generators for $C^0(\widehat{\Delta})$ as S-module can be read off the h-vector of Δ.

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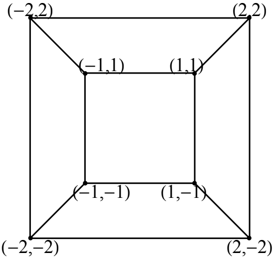

$\dim C^0_1(\mathcal{Q}) = 4$

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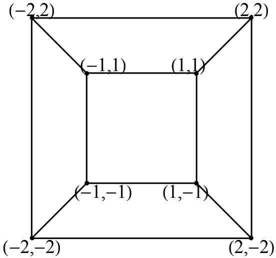

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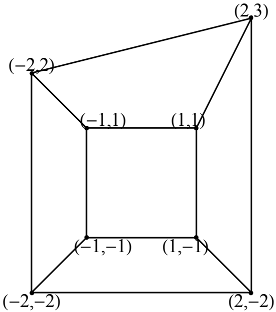

dim $C^0_1(\mathcal{Q}') = 3$

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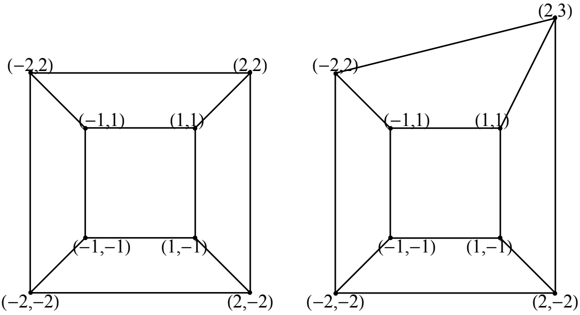

dim $C^0_1(\mathcal{Q}) = 4$ dim $C^0_1(\mathcal{Q}') = 3$

Relationship to polyhedral surfaces makes dim $C^0_1(\Delta)$ geometric in nature.

Comparing Perturbations

- $S = \mathbb{R}[x, y, z]$
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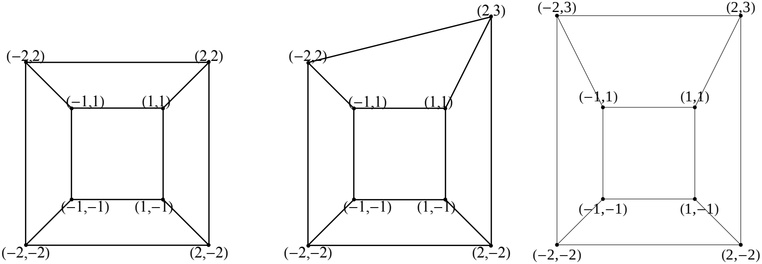

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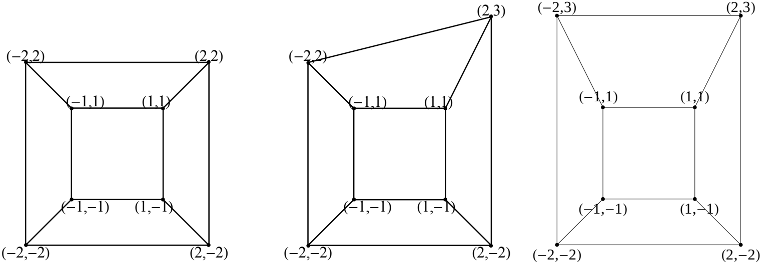

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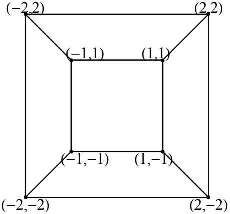

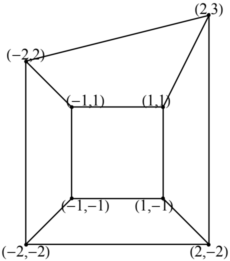

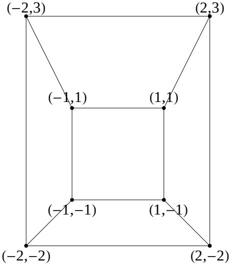

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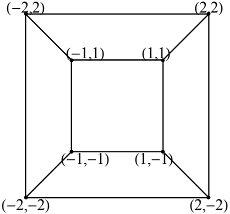

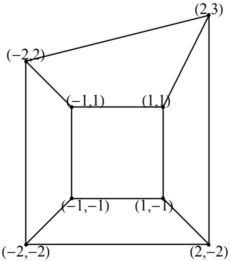

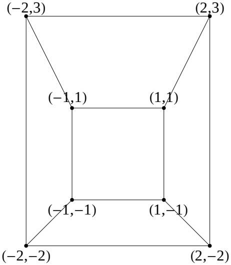

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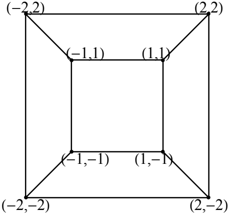

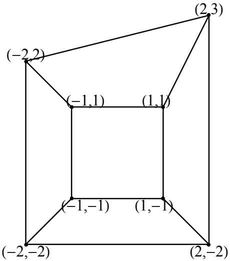

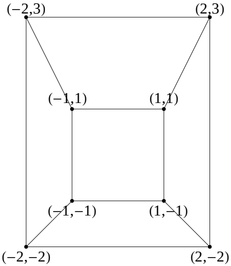

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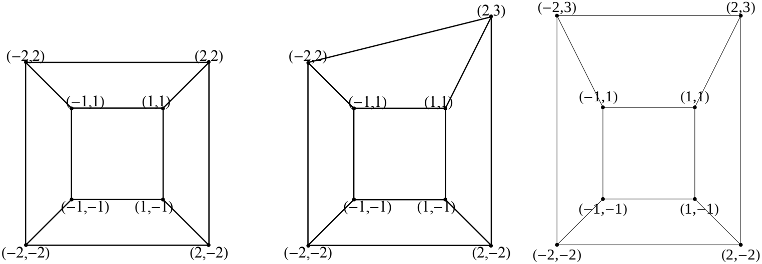

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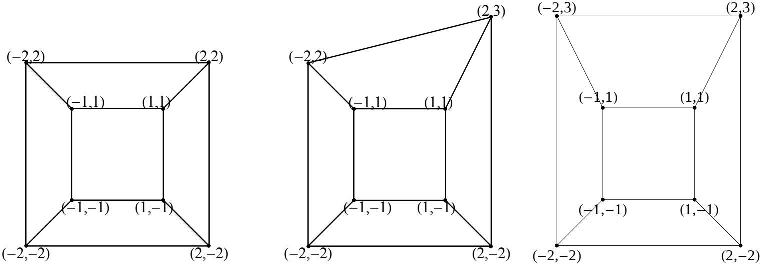

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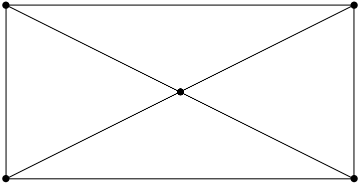

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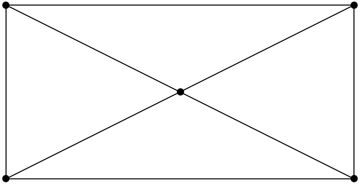

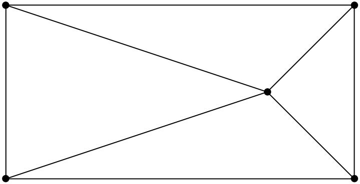

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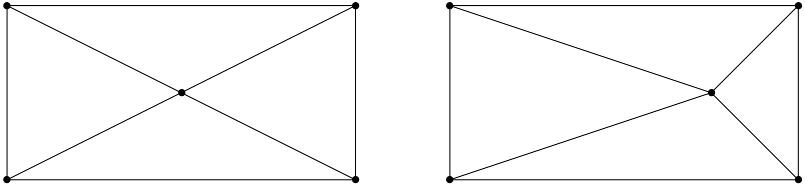

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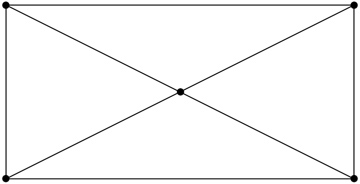

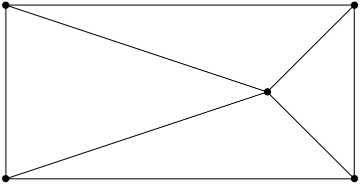

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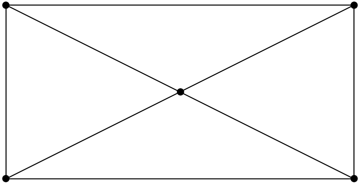

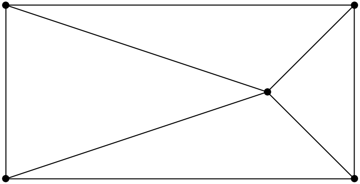

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Both $C^1(\widehat{\mathcal{T}})$ and $C^1(\widehat{\mathcal{T}'})$ are free S-modules.

Morgan-Scott triangulation

- $S = \mathbb{R}[x, y, z]$
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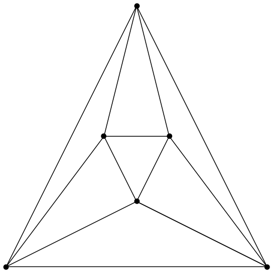

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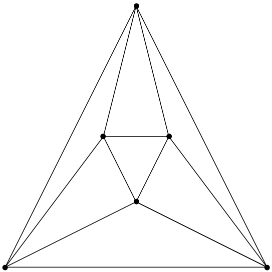

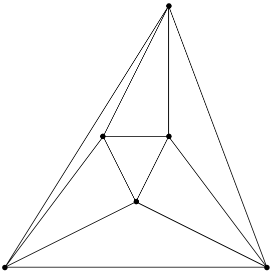

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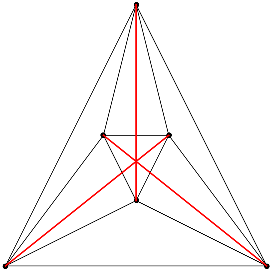

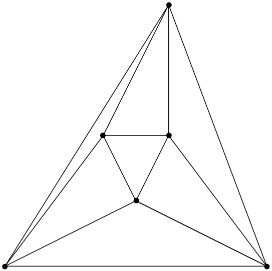

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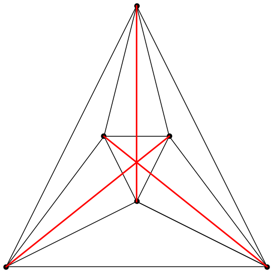

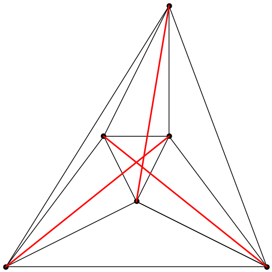

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Part II: Hilbert Polynomials and Regularity

Some Graded Commutative Algebra

Given a finitely generated graded $S = \mathbb{R}[x_1, \ldots, x_n]$-module M (like $C^r(\widehat{\Delta})$).

- $HF(M, d) := \dim M_d$ is the **Hilbert function** of M.

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- If $d >> 0$, $HF(M, d) = HP(M, d)$, where $HP(M, d)$ is the **Hilbert polynomial** of M.
- Upshot: $\dim C^r_d(\Delta) = \dim C^r(\widehat{\Delta})_d$ is eventually polynomial in d (in fact, linear combination of binomial coefficients)

The Good News and the Bad News

Good news: $HP(C^r(\widehat{\Delta}), d)$ has been computed for $\Delta \subset \mathbb{R}^2$.

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Bad news: $\dim C^r_d(\Delta)$ is still a mystery for small d.

- $\dim C^1_3(\Delta)$ still unknown for $\Delta \subset \mathbb{R}^2$!

Planar Hilbert Polynomials

Planar Simplicial Dimension [Alfeld-Schumaker '90]

If $\Delta \subset \mathbb{R}^2$ is a simply connected triangulation and $d \geq 3r+1$, then

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Conjecture [Schenck '97]

Above formula holds for $d \geq 2r+1$.

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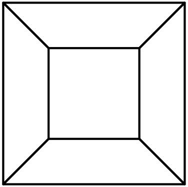

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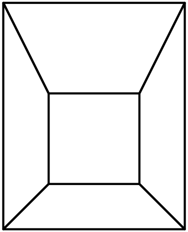

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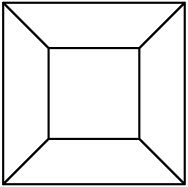

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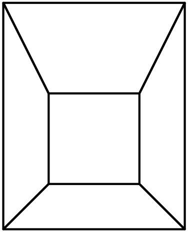

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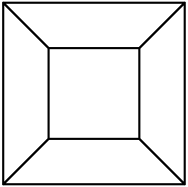

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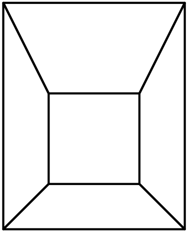

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In simplicial case, $d \geq 3r + 1$ suffices.

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Agreement of Hilbert Function and Polynomial [D. '14]

$\Delta \subset \mathbb{R}^2$ a planar polytopal complex. Let $F =$ maximum number of edges of a polygon of Δ. Then

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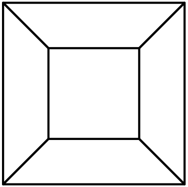

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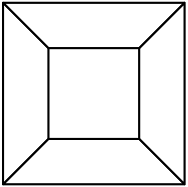

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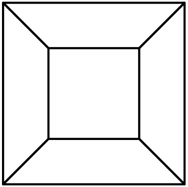

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Results on previous slide follow from bounding $reg(C^r(\widehat{\Delta}))$.

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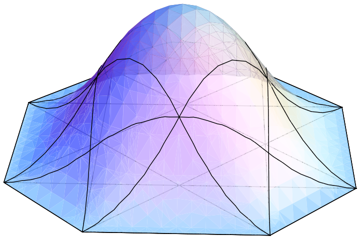
- Property 1 used to break bounding $reg(LS^{r,1}(\widehat{\Delta}))$ down into a local problem by fitting into exact complexes.
- Local problem solved directly

Thank You!



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