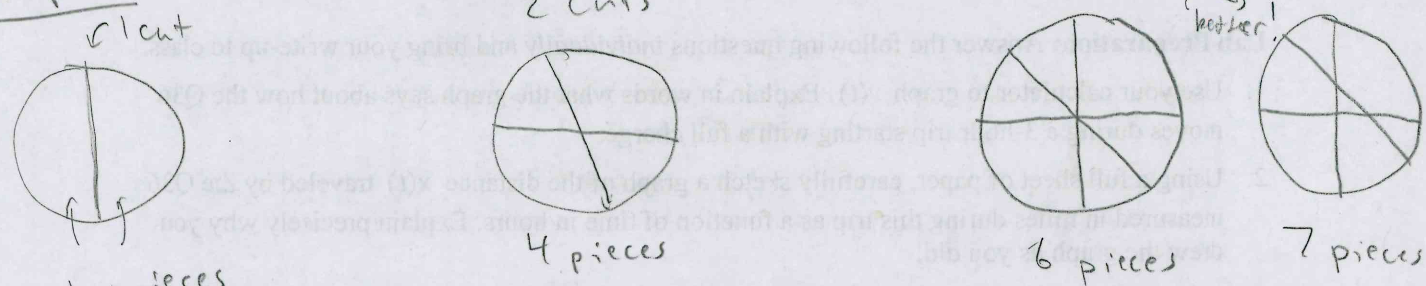
The Pizza Cutting Problem (Stillwater High School) 02/10/2017 1

Problem: If you are only allowed to make k straight-line cuts, what is the largest number of pizza "slices" you can make?

What about the largest number of pieces with no crust?

Example:



Give it a shot on your hand out!

Let P_k = largest # of pieces with k cuts

$P_1 = 2 \quad P_2 = 4 \quad P_3 = 7 \quad P_4 = 11 \quad P_5 = 16$

Pattern: $P_k = P_{k-1} + k = \cdots = 2 + 2 + 3 + 4 + \cdots + k$

Reason: Each added cut intersects previous $(k-1)$ lines, adding k new pieces.

What if you want to know $P_{1,520}$? Need formula!

$P_k = 1 + \boxed{1 + 2 + \cdots + k}$

Notice: $1 + 2 + \cdots + k$ $\Longrightarrow 2(1 + 2 + \cdots + k) = k(k+1)$

$+\; k + (k-1) + \cdots + 1$ $\qquad 1 + 2 + \cdots + k = \dfrac{k(k+1)}{2}$

$\underbrace{(k+1) + (k+1) + \cdots + (k+1)}_{k \text{ times!}}$

So, $P_k = 1 + \dfrac{k(k+1)}{2}$! (also $= 1 + k + \dfrac{k(k-1)}{2}$)

What about interior pieces?

$I_k = \#$ no crust pieces.

$I_1 = 0 \quad I_2 = 0 \quad I_3 = 1 \quad I_4 = 3 \quad I_5 = 6$.

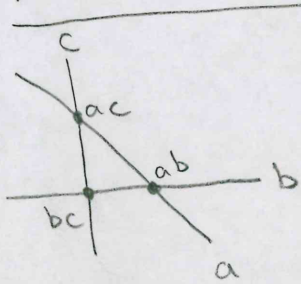
Pattern? $I_k = I_{k-1} + (k-2)$

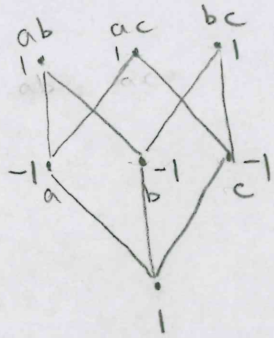
$$I_k = (k-2) + (k-1) + \dots + 1$$

$$= \frac{(k-1)(k-2)}{2} \qquad \text{[works for 1 \& 2 too!]}$$

$$\left(\text{also} = 1 - k + \frac{k(k-1)}{2}\right)$$

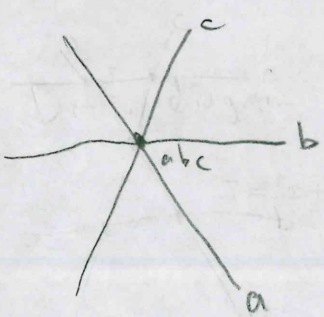
Different way to count (lets you count for any choice of cuts!)

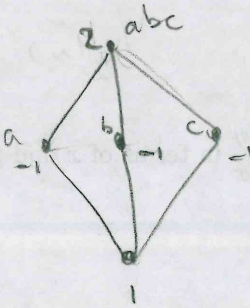




Add positive ones: 7

Just add: 1



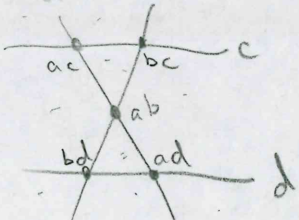


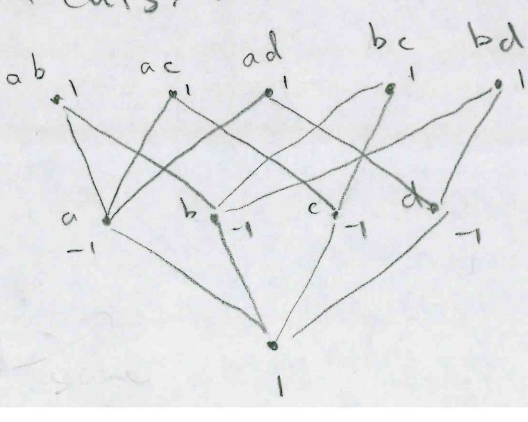
Add positive: 6

Just add: 0

Same

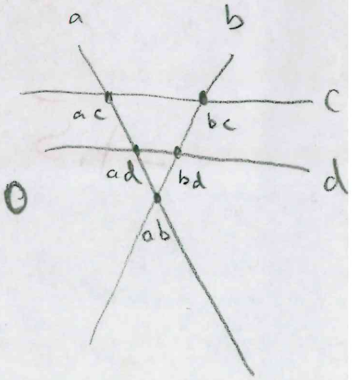
Draw it for 4 cuts! 5?





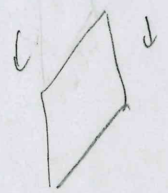
Add: 2

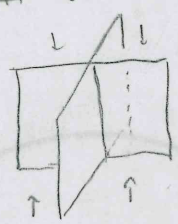
Add positive: 10

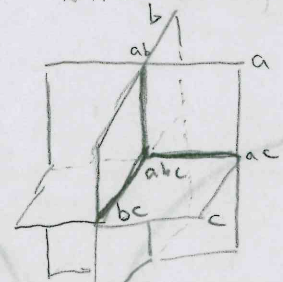


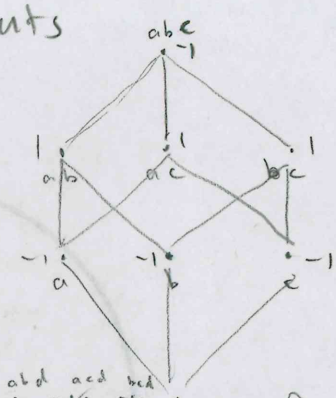
What about cutting 3D pizzas with planes? 02/10/2017

Let T_k = largest # of "chunks" with k straight cuts



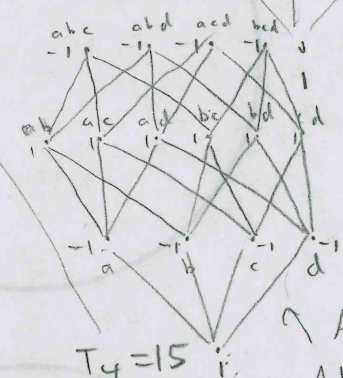






$T_1 = 2$ $T_2 = 4$ $T_3 = 8$

$P_1 = 2$ $P_2 = 4$ $P_3 = 7$



Add: 0

Add pos: 8

Add: -1?

Add pos: 15

$T_4 = 15$

Notice: $T_2 = T_1 + P_1$ $T_3 = T_2 + P_2$

In Fact: $T_k = T_{k-1} + P_{k-1}$!

Reason: New regions added can be counted by counting regions in new plane, which is divided by previous $k-1$ planes.

Formula: $$T_k = 1 + k + \frac{k(k-1)}{2} + \frac{k(k-1)(k-2)}{3}$$

Let N_k = # of crustless chunks

$$N_k = -1 + k - \frac{k(k-1)}{2} + \frac{k(k-1)(k-2)}{6}$$