Dimension of Tchebycheffian Spline Spaces on T-meshes

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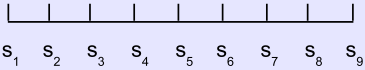
SIAM Conference on Applied Algebraic Geometry
August 3^{rd} 2017

T-meshes

Many bivariate spline spaces used in applications are based on the tensor-product of univariate spline spaces, and then are using a tensor-product mesh (B-splines, NURBS).



$\mathbb{S}_{p_h}$

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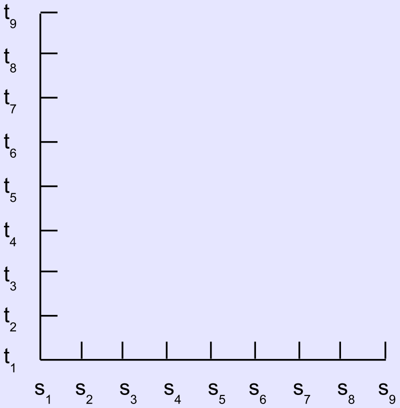

$\mathbb{S}_{p_h}$ $\mathbb{S}_{p_v}$

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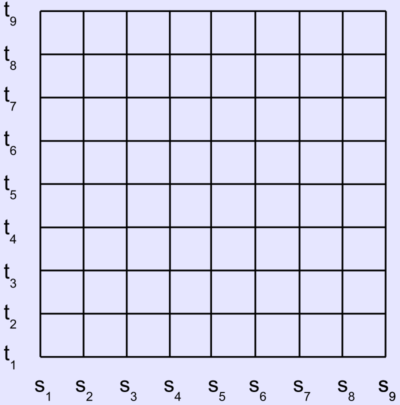
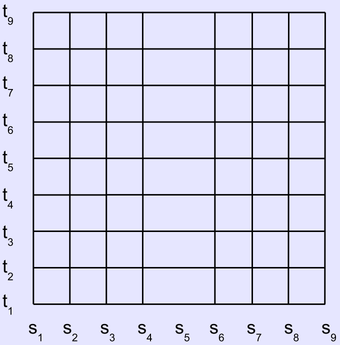
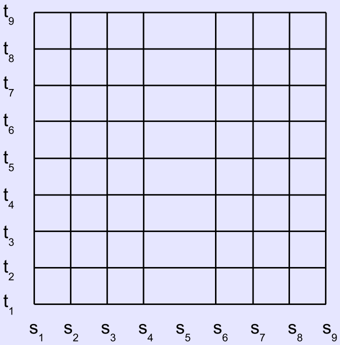


$\mathbb{S}_{\mathbf{p}} = \mathbb{S}_{p_h} \otimes \mathbb{S}_{p_v} = \{\text{polynomial in each cell, with suitable regularity}\}$

T-meshes



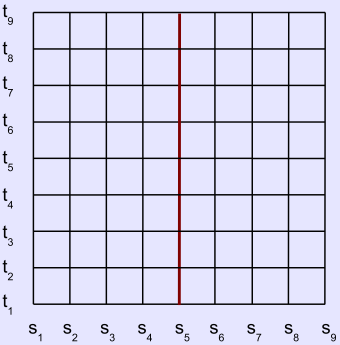
Tensor-product mesh



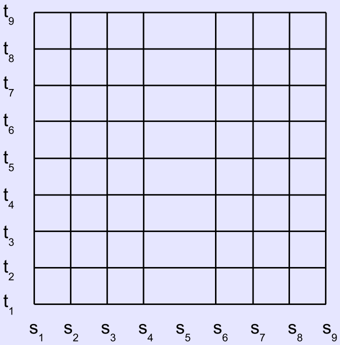
T-mesh

What refinement is allowed?

T-meshes



Tensor-product mesh



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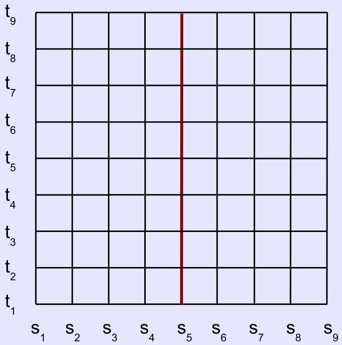
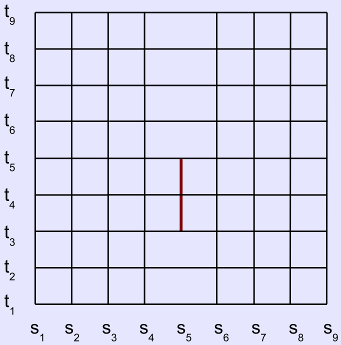


Tensor-product mesh



T-mesh

T-meshes allow local refinement!

T-meshes

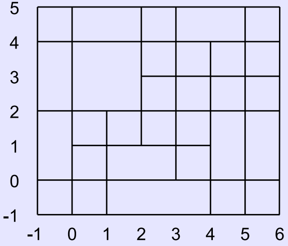
LOCAL REFINEMENT allows:

- adaptive algorithms
  - isogeometric analysis
  - data fitting
- lower dimension spaces (lower computational costs)
- handling unstructured data with a structured space
- ...and more...

T-meshes

Definition

A T-mesh $\mathcal{T}$ is a collection of axis-aligned rectangles $\mathcal{T}_2 = \{\sigma_i\}_{i=1}^{N_2}$ such that $\Omega \equiv \cup_i \sigma_i$ is connected and any pair of rectangles (*cells*) $\sigma_i, \sigma_j \in \mathcal{T}_2$ intersect each other only on their edges.



- $\mathcal{T}_1 = \mathcal{T}_1^h \cup \mathcal{T}_1^v$ = set of horizontal and vertical (closed) edges in $\bigcup_{\sigma \in \mathcal{T}_2} \partial\sigma$
- $\mathcal{T}_0 := \bigcup_{\tau \in \mathcal{T}_1} \partial\tau$ = set of vertices;

(Polynomial) splines spaces over T-meshes

A smoothness distribution on a T-mesh $\mathcal{T}$ is a map

$$\mathbf{r} : \mathcal{T}_1^o := \{\text{interior edges of } \mathcal{T}\} \longrightarrow \mathbb{N},$$

For any vertex $\gamma \in \mathcal{T}_0^o := \{\text{interior vertices of } \mathcal{T}\}$

$$r_h(\gamma) := r(\tau_v), \qquad r_v(\gamma) := r(\tau_h)$$

such that $\gamma = \tau_h \cap \tau_v$ and $\tau_h \in \mathcal{T}_1^{o,h}$, $\tau_v \in \mathcal{T}_1^{o,v}$.

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The *space of splines over a T-mesh $\mathcal{T}$ of bi-degree $\mathbf{p} = (p_h, p_v)$ and smoothness $\mathbf{r}$* is

$$\mathbb{S}_{\mathbf{p}}^{\mathbf{r}}(\mathcal{T}) := \left\{ s \in C^{\mathbf{r}}(\mathcal{T}) : s_{|\sigma} \in \mathbb{P}_{\mathbf{p}},\, \sigma \in \mathcal{T}_2 \right\}.$$

where $\mathbb{P}_{\mathbf{p}}$ is the space of polynomials of bi-degree $\mathbf{p}$, and we say that $f \in C^{\mathbf{r}}(\mathcal{T})$ if the partial derivatives of f up to order $r(\tau)$ are continuous across the edge τ, for $\tau \in \mathcal{T}_1^o$.

Extension to non-polynomial (Tchebycheffian) splines?

Motivations:

- exactly reproducing relevant shapes (cycloids, helices, transcendental curves, etc.)
- compared to NURBS, they can be used to reproduce the same shapes but with a better behaviour with respect to differentiation and integration

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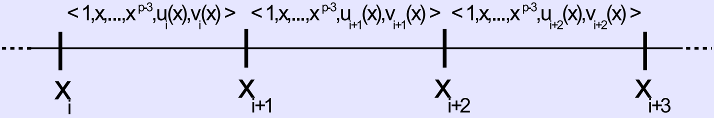
Tchebycheffian spline spaces are spaces of splines which, in each interval, belong to an Extended Tchebycheff space $\mathbb{T}_p([a, b])$ (ex: generalized splines).



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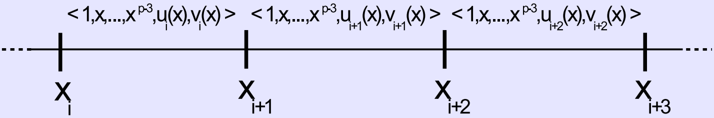
Tchebycheffian spline spaces are spaces of splines which, in each interval, belong to a space $\mathbb{T}_p([a,b])$ of dimension $p+1$ of functions defined on $[a,b]$ such that any Hermite interpolation problem with $p+1$ data in $[a,b]$ has a unique solution in $\mathbb{T}_p([a,b])$ (ex: generalized splines).



Spaces of Tchebycheffian splines over T-meshes

Let $\mathcal{T}$ be a T-mesh with a smoothness distribution $\mathbf{r}$, $\mathbf{p} := (p_h, p_v) \in \mathbb{N} \times \mathbb{N}$ with $p_h, p_v \geq 0$, and $\mathbf{T} := (T_h, T_v) := (\mathbb{T}^h_{p_h}, \mathbb{T}^v_{p_v})$. The *space of Tchebycheffian splines over the T-mesh* $\mathcal{T}$, denoted by $\mathbb{S}^{\mathbf{T},\mathbf{r}}_{\mathbf{p}}(\mathcal{T})$, is the space

$$\mathbb{S}^{\mathbf{T},\mathbf{r}}_{\mathbf{p}}(\mathcal{T}) := \left\{ s \in C^{\mathbf{r}}(\mathcal{T}) : s_{|\sigma} \in \mathbb{P}^{\mathbf{T}}_{\mathbf{p}},\, \sigma \in \mathcal{T}_2 \right\},$$

where

$$\mathbb{P}^{\mathbf{T}}_{\mathbf{p}} := \mathbb{T}^h_{p_h}([a_h, b_h]) \otimes \mathbb{T}^v_{p_v}([a_v, b_v]),$$

with $\mathbb{T}^h_{p_h}([a_h, b_h])$ and $\mathbb{T}^v_{p_v}([a_v, b_v])$ are two extended Tchebycheff spaces of dimension $p_h + 1$ and $p_v + 1$ respectively.

Dimension: the homological approach

The approach generalizes [Mourrain; 2014] (polynomial case).
We define the following subspaces of $\mathbb{P}_{\mathbf{p}}^{\mathbf{T}}$:

- for any vertical edge $\tau = \{\bar{x}\} \times [a_v, b_v]$

$$\mathbb{I}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\tau) := \{\, q \in \mathbb{P}_{\mathbf{p}}^{\mathbf{T}} : \; D_x^k q(\bar{x}, y) \equiv 0, \\ \forall y \in [a_v, b_v], \; k = 0, \dots, r(\tau) \,\},$$

- for any horizontal edge $\tau = [a_h, b_h] \times \{\bar{y}\}$

$$\mathbb{I}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\tau) := \{\, q \in \mathbb{P}_{\mathbf{p}}^{\mathbf{T}} : \; D_y^l q(x, \bar{y}) \equiv 0, \\ \forall x \in [a_h, b_h], \; l = 0, \dots, r(\tau) \,\},$$

- for any vertex $\gamma = (\bar{x}, \bar{y})$

$$\mathbb{I}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\gamma) := \{\, q \in \mathbb{P}_{\mathbf{p}}^{\mathbf{T}} : D_x^k D_y^l q(\bar{x}, \bar{y}) \equiv 0, \\ k = 0, \dots, r_h(\gamma), \;\; l = 0, \dots, r_v(\gamma) \,\}.$$

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Hermite interpolant assumption crucial for their dimension!

Dimension: the homological approach - basic idea

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
\mathfrak{I}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T}^o): & & 0 & \stackrel{\hat{\partial}_2}{\to} & \bigoplus_{\tau\in\mathcal{T}_1^o} \mathbb{I}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\tau) & \stackrel{\hat{\partial}_1}{\to} & \bigoplus_{\gamma\in\mathcal{T}_0^o} \mathbb{I}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\gamma) & \stackrel{\hat{\partial}_0}{\to} & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{P}_{\mathbf{p}}^{\mathbf{T}}(\mathcal{T}^o): 0 & \stackrel{\partial_3}{\to} & \bigoplus_{\sigma\in\mathcal{T}_2} \mathbb{P}_{\mathbf{p}}^{\mathbf{T}} & \stackrel{\partial_2}{\to} & \bigoplus_{\tau\in\mathcal{T}_1^o} \mathbb{P}_{\mathbf{p}}^{\mathbf{T}} & \stackrel{\partial_1}{\to} & \bigoplus_{\gamma\in\mathcal{T}_0^o} \mathbb{P}_{\mathbf{p}}^{\mathbf{T}} & \stackrel{\partial_0}{\to} & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\mathfrak{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T}^o): 0 & \stackrel{\bar{\partial}_3}{\to} & \bigoplus_{\sigma\in\mathcal{T}_2} \mathbb{P}_{\mathbf{p}}^{\mathbf{T}} & \stackrel{\bar{\partial}_2}{\to} & \bigoplus_{\tau\in\mathcal{T}_1^o} \mathbb{P}_{\mathbf{p}}^{\mathbf{T}}/\mathbb{I}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\tau) & \stackrel{\bar{\partial}_1}{\to} & \bigoplus_{\gamma\in\mathcal{T}_0^o} \mathbb{P}_{\mathbf{p}}^{\mathbf{T}}/\mathbb{I}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\gamma) & \stackrel{\bar{\partial}_0}{\to} & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

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$$\bigoplus_{\sigma \in \mathcal{T}_2} \mathbb{P}_{\mathbf{p}}^{\mathbf{T}} \xrightarrow{\partial_2} \bigoplus_{\tau \in \mathcal{T}_1^o} \mathbb{P}_{\mathbf{p}}^{\mathbf{T}}$$

$$[q_{\sigma_1}, q_{\sigma_2}, \ldots, q_{\sigma_{N_2}}] \longrightarrow [q_{\tau_1}, q_{\tau_2}, \ldots, q_{\tau_{N_1}}]$$

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$$[\ldots, q_{\sigma_i}, \ldots, q_{\sigma_j}, \ldots] \longrightarrow [\ldots, q_{\tau_k}, \ldots]$$

where σ_i and σ_j are the cells containing the edge τ_k.

Dimension: the homological approach - basic idea

$$\bigoplus_{\sigma \in \mathcal{T}_2} \mathbb{P}_{\mathbf{p}}^{\mathbf{T}} \xrightarrow{\partial_2} \bigoplus_{\tau \in \mathcal{T}_1^o} \mathbb{P}_{\mathbf{p}}^{\mathbf{T}}$$

$$[..., q_{\sigma_i}, ..., q_{\sigma_j}, ...] \longrightarrow [..., q_{\sigma_i} - q_{\sigma_j}, ...]$$

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Dimension: the homological approach - basic idea

$$\bigoplus_{\sigma \in \mathcal{T}_2} \mathbb{P}_{\mathbf{p}}^{\mathbf{T}} \xrightarrow{\partial_2} \bigoplus_{\tau \in \mathcal{T}_1^o} \mathbb{P}_{\mathbf{p}}^{\mathbf{T}}$$

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where σ_i and σ_j are the cells containing the edge τ_k.

The spline space can be written as

$$\begin{aligned}
\mathbb{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T}) &= \{q \in C^{\mathbf{r}}(\Omega) : q_\sigma := q|_\sigma \in \mathbb{P}_{\mathbf{p}}^{\mathbf{T}} \; \forall \sigma \in \mathcal{T}_2\} \\
&= \{[q_{\sigma_1}, \ldots, q_{\sigma_{N_2}}] \in \bigoplus_{\sigma\in\mathcal{T}_2} \mathbb{P}_{\mathbf{p}}^{\mathbf{T}} : q_{\sigma_i} - q_{\sigma_j} \in \mathbb{I}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\tau) \, \forall \tau \in \mathcal{T}_1^o\} \\
&= \ker(\bar{\partial}_2) = \mathbf{H}_2(\mathfrak{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T}^{\mathbf{o}}))
\end{aligned}$$

Similarly, the map ∂_1 is:

$$\bigoplus_{\tau \in \mathcal{T}_1^o} \mathbb{P}_{\mathbf{p}}^{\mathbf{T}} \xrightarrow{\partial_1} \bigoplus_{\gamma \in \mathcal{T}_0^o} \mathbb{P}_{\mathbf{p}}^{\mathbf{T}}$$

$$[q_{\tau_1}, q_{\tau_2}, \ldots, q_{\tau_{N_1}}] \longrightarrow [q_{\gamma_1}, q_{\gamma_2}, \ldots, q_{\gamma_{N_0}}]$$

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$$[\ldots, q_{\tau_{i_1}}, \ldots, q_{\tau_{i_2}}, \ldots, q_{\tau_{i_3}}, \ldots, q_{\tau_{i_4}}, \ldots] \longrightarrow [\ldots, q_{\gamma_k}, \ldots]$$

where τ_{i_1}, τ_{i_2} are the horizontal edges having γ_k as endpoint, and τ_{i_3}, τ_{i_4} are the vertical edges having γ_k as endpoint.

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$$[\ldots, q_{\tau_{i_1}}, \ldots, q_{\tau_{i_2}}, \ldots, q_{\tau_{i_3}}, \ldots, q_{\tau_{i_4}}, \ldots] \longrightarrow [\ldots, q_{\tau_{i_1}} - q_{\tau_{i_2}} + q_{\tau_{i_3}} - q_{\tau_{i_4}}, \ldots]$$

where τ_{i_1}, τ_{i_2} are the horizontal edges having γ_k as endpoint, and τ_{i_3}, τ_{i_4} are the vertical edges having γ_k as endpoint.

Computing the dimension

Considering the Euler characteristic of $\mathfrak{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T}^o)$, we get

$$\dim\Big(\bigoplus_{\sigma\in\mathcal{T}_2}\mathbb{P}_{\mathbf{p}}^{\mathbf{T}}\Big) - \dim\Big(\bigoplus_{\tau\in\mathcal{T}_1^o}\mathbb{P}_{\mathbf{p}}^{\mathbf{T}}/\mathbb{I}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\tau)\Big) + \dim\Big(\bigoplus_{\gamma\in\mathcal{T}_0^o}\mathbb{P}_{\mathbf{p}}^{\mathbf{T}}/\mathbb{I}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\gamma)\Big)$$
$$= \dim\big(H_2(\mathfrak{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T}^o))\big) - \dim\big(H_1(\mathfrak{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T}^o))\big) + \dim\big(H_0(\mathfrak{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T}^o))\big).$$

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$$=\dim\big(H_2(\mathfrak{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T}^o))\big)-\dim\big(H_1(\mathfrak{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T}^o))\big)+0.$$

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$$=\dim\big(\mathbb{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T})\big)-\dim\big(H_0(\mathfrak{I}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T}^o))\big)+0.$$

Theorem (B., Lyche, Manni, Roman, Speleers)

$$\begin{aligned}\dim\left(\mathbb{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T})\right) &= \sum_{\sigma\in\mathcal{T}_2}(p_h+1)(p_v+1) \\ &- \sum_{\tau\in\mathcal{T}_1^{o,h}}(p_h+1)(\mathbf{r}(\tau)+1) \\ &- \sum_{\tau\in\mathcal{T}_1^{o,v}}(\mathbf{r}(\tau)+1)(p_v+1) \\ &+ \sum_{\gamma\in\mathcal{T}_0^{o}}(r_h(\gamma)+1)(r_v(\gamma)+1) + D,\end{aligned}$$

where

$$D := \dim\left(H_0(\mathfrak{I}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T}^o))\right).$$

Theorem (B., Lyche, Manni, Roman, Speleers)

$$\dim\left(\mathbb{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T})\right) = \sum_{\sigma\in\mathcal{T}_2}(p_h+1)(p_v+1) - \sum_{\tau\in\mathcal{T}_1^{o,h}}(p_h+1)(\mathbf{r}(\tau)+1) - \sum_{\tau\in\mathcal{T}_1^{o,v}}(\mathbf{r}(\tau)+1)(p_v+1) + \sum_{\gamma\in\mathcal{T}_0^{o}}(r_h(\gamma)+1)(r_v(\gamma)+1) + D,$$

where

$$D := \dim\left(H_0(\mathfrak{I}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T}^o))\right).$$

Can we bound D in a meaningful way?

Bounding D

Given $\mathbf{d} := (d_1, ..., d_m)$, $0 \le d_i \le p$, $d_i \in \mathbb{N}$, $i = 1, ..., m$, an Extended Tchebycheff space $\mathbb{T}_p$ of dimension $p+1$ on $[a,b]$ has the **d**-sum property if for any m distinct points $x_1, ..., x_m \in [a,b]$

$$\dim\Big(\sum_{i=1}^{m} \mathbb{I}_{\mathbf{p}}^{\mathbf{T},d_i}(x_i)\Big) = \min\Big(p+1, \sum_{i=1}^{m} p - d_i\Big),$$

$$\mathbb{I}^{\mathbf{T}_p,d_i}(x_i) := \{q \in \boldsymbol{t}_p : D^l q(x_i) = 0,\ l = 0, ..., d_i\}.$$

Hermite interp. assumption not enough!

Bounding D

Given $\mathbf{d} := (d_1, ..., d_m)$, $0 \le d_i \le p$, $d_i \in \mathbb{N}$, $i = 1, ..., m$, an Extended Tchebycheff space $\mathbb{T}_p$ of dimension $p + 1$ on $[a, b]$ has the **d**-sum property if for any m distinct points $x_1, ..., x_m \in [a, b]$

$$\dim\Big(\sum_{i=1}^{m} \mathbb{I}_{\mathbf{p}}^{\mathbf{T}, d_i}(x_i)\Big) = \min\Big(p + 1, \sum_{i=1}^{m} p - d_i\Big),$$

$$\mathbb{I}^{\mathbf{T}_p, d_i}(x_i) := \{q \in \boldsymbol{t}_p : D^l q(x_i) = 0,\ l = 0, ..., d_i\}.$$

Hermite interp. assumption not enough!

Theorem (B., Lyche, Manni, Speleers)

An Extended Complete Tchebycheff space (ECT space) $\mathbb{T}_p$, that is, spanned by a set of functions $\{u_0, ..., u_p\}$ such that

$$\text{det}(\text{Hermite collocation matrix of } u_0, ..., u_p \text{ at } x_0, ..., x_k) > 0,$$

*for any $x_0 \le ... \le x_k$, $k = 0, ..., p$, satisfies the **d**-sum property for any $\mathbf{d} := (d_1, ..., d_m)$, $0 \le d_i \le p$, $d_i \in \mathbb{N}$, $i = 1, ..., m$ and any m.*

Idea of the proof

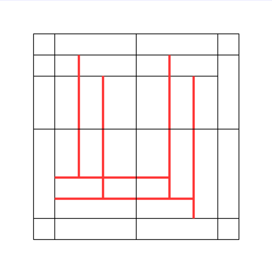
Use the generalized power basis

- It exists if and only if $\mathbb{T}_p$ is an ECT
- mimics the properties of the monomial basis$\{(x-c)^k/k!\}_{k=0,\dots,p}$ (derivatives at c)
- write everything in this basis

Bounding D

A segment ρ composed of edges of $\mathcal{T}_1^o$ which cannot be extended by adding other edges of $\mathcal{T}_1^o$ and does not intersect the boundary of the T-mesh, is a maximal interior segment.

$$\mathrm{MIS}_h(\mathcal{T}) := \{\text{horizontal maximal interior segments}\},$$
$$\mathrm{MIS}_v(\mathcal{T}) := \{\text{vertical maximal interior segments}\},$$
$$\mathrm{MIS}(\mathcal{T}) := \mathrm{MIS}_h(\mathcal{T}) \cup \mathrm{MIS}_v(\mathcal{T}).$$



MIS highlighted in red.

Bounding D

Given an ordering ι of $\mathrm{MIS}(\mathcal{T})$, for any $\rho \in \mathrm{MIS}(\mathcal{T})$, we denote by $\Gamma_\iota(\rho)$ the set of vertices of ρ which do not belong to $\rho' \in \mathrm{MIS}(\mathcal{T})$ with $\iota(\rho') > \iota(\rho)$. For any $\rho \in \mathrm{MIS}(\mathcal{T})$ we define its weight

$$\omega_\iota(\rho) := \begin{cases} \sum_{\gamma \in \Gamma_\iota(\rho)} (p_h - r_h(\gamma)), & \text{if } \rho \in \mathrm{MIS}_h(\mathcal{T}) \\ \sum_{\gamma \in \Gamma_\iota(\rho)} (p_v - r_v(\gamma)), & \text{if } \rho \in \mathrm{MIS}_v(\mathcal{T}) \end{cases}.$$

Theorem (B., Lyche, Manni, Roman, Speleers)

If ι is an ordering of $\mathrm{MIS}(\mathcal{T})$, and $\mathbb{S}_{\mathbf{p}}^{\mathbf{T},\mathbf{r}}(\mathcal{T})$ is an Extended Tchebycheff spline space with $\mathbf{T} = (\mathbb{T}^h_{p_h}, \mathbb{T}^v_{p_v})$ being a couple of ECT spaces, then

$$0 \le D \le \sum_{\rho \in \mathrm{MIS}_h(\mathcal{T})} (p_h + 1 - \omega_\iota(\rho))_+ (p_v - r(\rho)) + \sum_{\rho \in \mathrm{MIS}_v(\mathcal{T})} (p_h - r(\rho)) (p_v + 1 - \omega_\iota(\rho))_+,$$

How to get T-meshes for which $D = 0$?

Algorithm (generalizes [Mourrain;2014]) For each new edge:

- insert the new edge τ
- if τ does not extend an existing edge, then extend it so that so that the horizontal (vertical) maximal segment containing τ, say $\rho(\tau)$, intersects Ω or satisfies $\omega_\iota(\rho(\tau)) \geq p_h + 1$ $(\omega_\iota(\rho(\tau)) \geq p_v + 1)$

If you start from a T-mesh with $\omega_\iota(\rho) \geq p_h + 1$ for any $\rho \in \mathrm{MIS}_h(\mathcal{T})$ and $\omega_\iota(\rho) \geq p_v + 1$ for any $\rho \in \mathrm{MIS}_v(\mathcal{T})$, such property is preserved by the algorithm.

$\Longrightarrow$ The algorithm always gives for which $D = 0$

How to get T-meshes for which $D = 0$?

Definition (Cycle (of MIS))

A sequence $\rho_1, \ldots, \rho_n$ of composite edges (maximal interior segments) in a T-mesh forms a cycle (of MIS) if each ρ_i has one of its endpoints in the interior of ρ_{i+1} ($\rho_{n+1} := \rho_1$).

How to get T-meshes for which $D = 0$?

Definition (Cycle (of MIS))

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Sufficient conditions which avoid extending inserted edges:

- T-meshes **without cycles** and $p_h \geq 2r(\tau) + 1$ for all $\tau \in \mathcal{T}_1^{o,v}$ and $p_v \geq 2r(\tau) + 1$ for all $\tau \in \mathcal{T}_1^{o,h}$
- T-meshes **without cycles of MIS** and $p_h \geq 2r(\tau) + 1$ for all $\tau \in \mathcal{T}_1^{o,v}$ and $p_v \geq 2r(\tau) + 1$ for all $\tau \in \mathcal{T}_1^{o,h}$
- **hierarchical** T-meshes (the T-mesh is obtained by repeated refinement of a tensor-product mesh) and $p_h \geq 2r(\tau) + 1$ for all $\tau \in \mathcal{T}_1^{o,v}$ and $p_v \geq 2r(\tau) + 1$ for all $\tau \in \mathcal{T}_1^{o,h}$

Concluding remarks

- The results of polynomial and Tchebycheffian spline spaces are very similar (identical under the assumption we made)
- Key points: ET spaces and ECT spaces (implying $\mathbf{d} - \mathit{sum}$ property) assumptions
- The main spaces used for nonpolynomial spline are ECT (trigonometric, hyperbolic)
- Knowing the dimension of the space is the first step to construct an efficient basis

Concluding remarks

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Thank you for your kind attention!

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