Smooth Splines on Surfaces with General Topology

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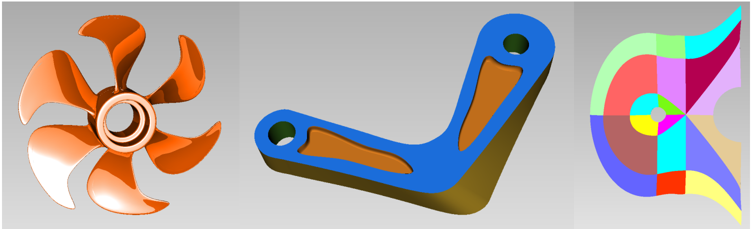


Motivations

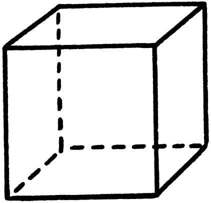
☞ Shapes are not always rectangular.

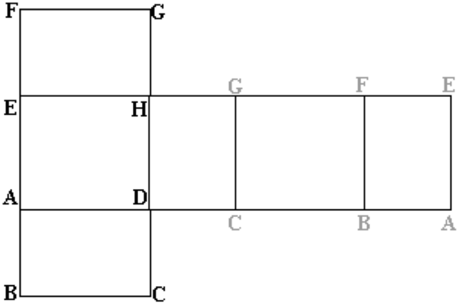
☞ Uniform description of shapes: avoid trimmed patches, impose regularity conditions across edges.

☞ Shape representation resistant to deformation: optimisation, fitting, isogeometric analysis, . . .

A topological structure

- A **polyhedral complex** $\mathscr{M}$: **cells** $\sigma \in \mathscr{M}_2$, shared **edges** $\tau \in \mathscr{M}_1$, shared **vertices** $\gamma \in \mathscr{M}_0$.





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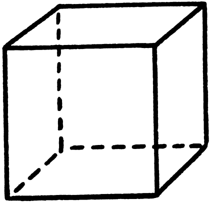

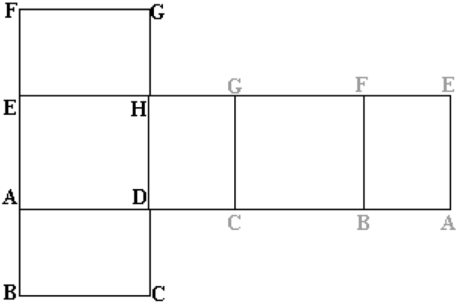

We will consider □ rectangular cells and Δ triangular cells.

A sheaf structure

- A poset $L = \{(\delta, \sigma) \text{ with } \sigma \in \mathscr{M}_2, \delta \in \partial\sigma\}$ with $(\delta, \sigma) < (\delta', \sigma')$ if $\delta \supset \delta'$ and $\delta' \subset \sigma \cap \sigma'$.
- **Rings of functions:** for $(\delta, \sigma) \in L$, $\mathcal{F}(\delta, \sigma) = \mathcal{R}/\mathcal{I}^{r+1}(\delta, \sigma)$ where $\mathcal{I}^{r+1}(\tau, \sigma) = (l_\tau^{r+1})$ and $\mathcal{I}^{r+1}(\gamma, \sigma) = \sum_{(\tau', \sigma') < (\gamma, \sigma)} \rho^*_{\sigma,\gamma}(l_\tau^{r+1})$.
- **Restriction maps:**
  $\rho_{(\delta,\sigma),(\delta',\sigma')} : f \in \mathcal{F}/\mathcal{I}^{r+1}(\delta, \sigma) \to f \circ \rho_{\sigma,\sigma'} \in \mathcal{F}/\mathcal{I}^{r+1}(\delta', \sigma')$.
  Transition maps:
  $\rho_{(\tau,\sigma),(\tau,\sigma')} : f \in \mathcal{F}/\mathcal{I}^{r+1}(\tau, \sigma) \to f \circ \phi_{\sigma,\sigma'} \in \mathcal{F}/\mathcal{I}^{r+1}(\tau, \sigma')$.

☞ **Global sections $\Gamma(L, \mathcal{F}) \Leftrightarrow$ Splines of $\mathscr{M}$**

Studied cases:

- $\mathcal{R} = \begin{cases} \mathbb{R}[u, v] \\ \text{Splines on } [0,1]^2 \text{ with nodes at } u = \frac{1}{2},\ v = \frac{1}{2}. \end{cases}$
- Regularity: $r = 1$

Splines on $\mathscr{M}$

Definition (Regularity)

A polynomial vector $[f_\sigma]_{\sigma \in \mathscr{M}_2}$ is C^r across the edge τ shared by σ , σ' iff

$$J^r(f_\sigma) = J^r(f_{\sigma'} \circ \phi_{\sigma',\sigma}) \text{ around } \tau.$$

Equivalently: $f_\sigma - f_{\sigma'} \circ \phi_{\sigma',\sigma} \in I^{r+1}(\tau, \sigma)$

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Definition ($\mathcal{S}_k^r(\mathscr{M})$ spline space over $\mathscr{M}$)

A spline f of "degree" $\leq k$ and regularity C^r over $\mathscr{M}$ is given by a collection of polynomials $f = [f_\sigma]_{\sigma \in \mathscr{M}_2}$ of degree (resp. bi-degree) $\leq k$ (resp. $\leq (k,k)$) on triangles (resp. rectangles), which are "C^r across the edge.

Transition maps (r=1)

$$\phi_{\sigma_2,\sigma_1} : (u_1, v_1) \longrightarrow (u_2, v_2) = \begin{pmatrix} v_1\, \mathfrak{b}_{\tau,\gamma}(u_1) + \mathcal{O}(v_1^2) \\ u_1 + v_1\, \mathfrak{a}_{\tau,\gamma}(u_1) + \mathcal{O}(v_1^2) \end{pmatrix}$$

We assume rational maps: $\mathfrak{a}(u) = \frac{a(u)}{c(u)}$, $\mathfrak{b}(u) = \frac{b(u)}{c(u)}$, with $a, b, c \in \mathbb{R}[u]$.
☞ **Glueing data:** $[a(u), b(u), c(u)]$.

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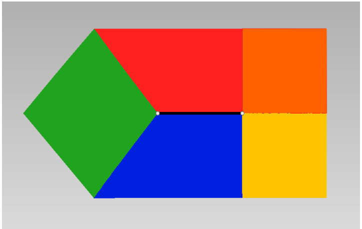
Example:



$$\phi_{\sigma',\sigma}(u, v) = (-v + \mathcal{O}(v^2), u + v(\delta(u)cos(\frac{2\pi}{n_\gamma}) - \delta'(u)cos(\frac{2\pi}{n_{\gamma'}})) + \mathcal{O}(v^2)).$$

Compatibility conditions

- $J_r(\phi_{\sigma_1,\sigma_n}) \circ \cdots \circ J(\phi_{\sigma_2,\sigma_1}) = Id.$

$$\prod_{i=1}^{F} \begin{pmatrix} 0 & 1 \\ \mathfrak{b}_i(0) & \mathfrak{a}_i(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Example: construction from a fan $\mathbf{u}_1, \ldots, \mathbf{u}_{F'} \in \mathbb{R}^2$ around $\gamma = (0,0)$:

$$\mathbf{u}_{i-1} = \mathfrak{a}_i(0)\mathbf{u}_i + \mathfrak{b}_i(0)\,\mathbf{u}_{i+1} \tag{1}$$

- *Ample space splines*: at every point γ of a face σ of $\mathscr{M}$, the space of values and differentials at γ, namely $\left[\, f(\gamma),\, \partial_{u_\sigma}(f)(\gamma),\, \partial_{v_\sigma} f(\gamma) \,\right]$ for $f \in \mathcal{S}_k^r(\mathscr{M})$, is of dimension 3.
  $\mathfrak{a}_i(0) = 0$ for all edges $\Rightarrow$ constraints between $\mathfrak{a}_i'(0), \mathfrak{b}_i(0), \mathfrak{b}_i'(0)$.
- *Topological restriction.* Preserve orientation across edges, $\mathfrak{b}(u) < 0$.

G^1-splines across an edge τ

If τ defined by $v = 0$ on σ_1, $u = 0$ on σ_2, (f_1, f_2) **G^1-regular across τ** iff

$$J^1(f_2 \circ \phi_{\sigma_2,\sigma_1}) = J^1(f_1)$$

iff

$$\begin{pmatrix} f_1(u_1, v_1) \\ f_2(u_2, v_2) \end{pmatrix} = \begin{pmatrix} c + \int_0^{u_1} A(t)dt & - v_1\, C(u_1) & + v_1^2\, K_1(u_1, v_1) \\ c + \int_0^{v_2} A(t)dt & - u_2\, B(v_2) & + u_2^2 K_2(u_2, v_2) \end{pmatrix}$$

with $\qquad a(u)A(u) + b(u)B(u) + c(u)C(u) = 0.$

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Syzygies:

$$\mathrm{Syz}_{\mathcal{R}}(a, b, c) = \{[A, B, C] \in \mathcal{R}^3 \mid a(u)A(u) + b(u)B(u) + c(u)C(u) = 0\}$$

1. $\mathrm{Syz}(a, b, c)$ is a free $\mathbb{R}[u_1]$-module of rank 2.
2. It is generated by vectors (A_1, B_1, C_1), (A_2, B_2, C_2) of coefficient degree μ and $\nu = n - \mu + 1 + F_\Delta - e - 2m$ where μ is the smallest possible coefficient degree, $n = \max\big(\deg(a), \deg(b), \deg(c)\big)$, $e = \min(n - \deg(a) + 1, n - \deg(b) + F_\Delta(\sigma_2), n - deg(c) + F_\Delta(\sigma_1)$, $m = \min\big(F_\Delta(\sigma_1), F_\Delta(\sigma_2)\big)$.
3. For $k \in \mathbb{N}$, the dimension of $\mathrm{Syz}(a, b, c)_k$ as a vector space over $\mathbb{R}$ is

$$d_\tau(k) = (k - \mu - m + 1)_+ + (k - n + \mu + m - F_\Delta + e)_+$$

   where $t_+ = \max(0, t)$ for $t \in \mathbb{Z}$.
4. The generators (A_1, B_1, C_1), (A_2, B_2, C_2) of $\mathrm{Syz}(a, b, c)$ can be chosen so that

$$(a, b, c) = (B_1 C_2 - B_2 C_1, C_1 A_2 - C_2 A_1, A_1 B_2 - A_2 B_1).$$

   (Hilbert-Burch theorem)

Syzygies of splines

$\mathcal{U}^r := \{g : [0,1] \to \mathbb{R} \; s.t. \; g_{|[0,\frac{1}{2}]}, g_{|[\frac{1}{2},1]} \in \mathbb{R}[u] \text{ and } g \text{ is } C^r \text{ at } \frac{1}{2}\}$

For $a = [a_1, a_2], b = [b_1, b_2], c = [c_1, c_2] \in \mathcal{U}^r$, we have the exact sequence:

$$0 \longrightarrow \mathrm{Syz}_{\mathcal{U}^r}(a, b, c) \longrightarrow \mathrm{Syz}(a_1, b_1, c_1) \times \mathrm{Syz}(a_2, b_2, c_2) \xrightarrow{\phi} \mathcal{Q}^r \times \mathcal{Q}^r \times \mathcal{Q}^r \xrightarrow{\psi} \mathcal{Q}^r \longrightarrow 0$$

where $\mathcal{Q}^r = \mathbb{R}[u]/(1-2u)^{r+1}$.

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where $\mathcal{Q}^r = \mathbb{R}[u]/(1-2u)^{r+1}$.

- Generators as an $\mathbb{R}[u]$-module:

$$\{(0,(2u-1)^{r+1}p_2), (0,(2u-1)^{r+1}q_2), (\tilde{p}_2, p_2), (\tilde{q}_2, q_2),$$
$$((2u-1)^{r+1}q_1, 0), ((2u-1)^{r+1}p_1, 0)\}$$

  where p_1, q_1 (*resp.* p_2, q_2) are free generators of Syz_1 (resp. Syz_2).
- Dimension formula:

$$\tilde{d}_\tau(k,r) = \dim(\mathrm{Syz}_k^r) = (k-\mu_1+1)_+ + (k-n_1+\mu_1+e_1)_+ +$$
$$(k-\mu_2+1)_+ + (k-n_2+\mu_2+e_2)_+ - \min(r+1,k) - (r+1).$$

Dimension formula for G^1-splines

Theorem

For $k \geq \mathfrak{s}^*$,

$$\begin{aligned}\dim \mathcal{S}_k^1(\mathscr{M}) &= (k-3)^2 F_\square + \tfrac{1}{2}(k-4)(k-5)F_\Delta \\ &+ \textstyle\sum_{\tau \in \mathscr{M}_1} d_\tau(k) + 4F_\square + 3F_\Delta - 9F_1 + 3F_0 + F_+\end{aligned}$$

where

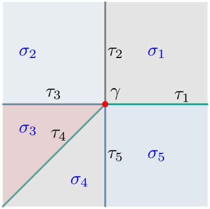
- *$F_\square$ is the number of quadrangular cells, F_Δ is the number of triangular cells,*
- *F_1 is the number of edges,*
- *F_0 (resp. F_+) is the number of (resp. crossing) vertices.*

For splines in $\mathcal{U}_k^r \times \mathcal{U}_k^r$,

$$\begin{aligned}\dim \mathcal{S}_{k,r}^1(\mathscr{M}) &= (2k-r-3)^2 F_2 \\ &+ \textstyle\sum_{\tau \in \mathscr{M}_1} \tilde{d}_\tau(k,r) + 4F_2 - 9F_1 + 3F_0 + F_+\end{aligned}$$

Crossing edges: For $\gamma \in \mathscr{M}_0$ and $\sigma_1, \ldots, \sigma_{N(\gamma)}$ the $N(\gamma)$ cells around γ

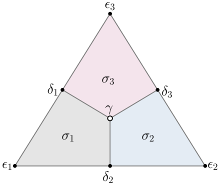
- $\mathfrak{c}_\tau(\gamma) = 1$ if τ is a **crossing edge** ($\mathfrak{a}(0) = 0$) and 0 otherwise.
- $\mathfrak{c}_+(\gamma) = 1$ if all edges are crossing edges and 0 otherwise.



For $k \geq \mathfrak{s}$,

- **Vertex functions at γ:** $3 + F(\gamma) - \sum_{\tau \ni \gamma} \mathfrak{c}_\tau(\gamma) + \mathfrak{c}_+(\gamma)$
- **Edge functions at τ:** $d_k(\tau) - 9 + \mathfrak{c}_\tau(\gamma) + \mathfrak{c}_\tau(\gamma')$,
- **Face functions at σ:** $(k-3)^2$ if $\square$, $\frac{1}{2}(k-4)(k-5)$ if Δ

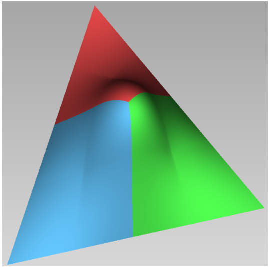
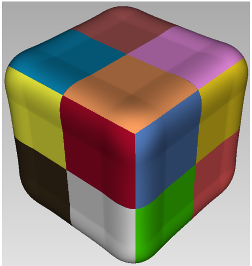
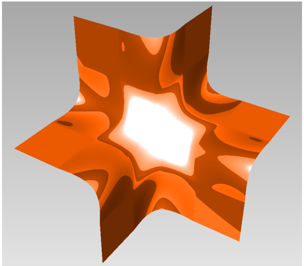
A round corner



Symmetric gluing data at γ and at the crossing boundary vertices δ_i.
Transition maps across the interior edge τ_i: $[a, b, c] = [(u-1), -1, 1]$.
Syzygies generators: $Z_1 = [0, 1, 1], Z_2 = [1, u, 1]$.
Dimension of $\mathcal{S}_4^1$: 48.
The number of basis functions attached to

- γ: $6 = 1 + 2 + 3$.
- δ_i: $4 = 1 + 2 + 2 - 1$.
- ϵ_i: $4 = 1 + 2 + 1$.
- the interior edge τ_i: $2 \times 4 - 7 = 1$.
- the boundary edges: $2(4 - 3) = 2$.
- a face σ_i: $(4 - 3)^2 = 1$.

Examples of bi-cubic G^1 surfaces

Open questions

- Dimension formula and basis for lower degree splines.
- Extension to G^k splines.
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- Extension to general planar subdivisions, volume subdivisions.
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Thank you for your attention.