Subdivision and spline spaces

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set up

- Δ is a k-dimensional simplicial complex in $\mathbb{R}^k$
- Δ is modified by subdividing a single maximal cell $\sigma \in \Delta_k$ to obtain Δ'
- Δ'' a subdivision of σ
- Δ' is a complex if any modifications made to the boundary of σ occur only on $\sigma \cap \partial(\Delta)$
- how do relate splines on a simplicial complex Δ and Δ'' to splines on a complex Δ'?

main definitions: $\mathcal{R}/\mathcal{J}(\Delta)$

$$0 \longrightarrow \bigoplus_{\sigma\in\Delta_k} R \xrightarrow{\partial_k} \bigoplus_{\tau\in\Delta^0_{k-1}} R/J_\tau \xrightarrow{\partial_{k-1}} \bigoplus_{\psi\in\Delta^0_{k-2}} R/J_\psi \xrightarrow{\partial_{k-2}} \dots \xrightarrow{\partial_1} \bigoplus_{v\in\Delta^0_0} R/J_v \longrightarrow 0,$$

where for an interior i-face $\gamma \in \Delta^0_i$,

$$J_\gamma = \langle l_{\hat{\tau}}^{r+1} \mid \gamma \subseteq \tau \in \Delta_{k-1}\rangle$$

- complex of $R = \mathbb{R}[x_0, \dots, x_k]$ modules
- ∂_i the usual boundary operator in relative homology
- Δ_i the set of i-dimensional faces
- Δ^0_i the set of interior i-dimensional faces
- all k-dimensional faces are considered interior so $\Delta_k = \Delta^0_k$

main definitions: simple and split subdivisions

$\sigma \in \Delta_k$, and Δ'' a subdivision of σ

$$\partial(\sigma) = \partial(\Delta'') \quad \text{on} \quad \Delta^0$$

Then the resulting subdivision Δ' is again a simplicial complex, and we call the subdivision a **simple** subdivision.

A simple subdivision Δ' is called **split** if for every $\gamma \in \partial(\Delta'')_i$ but not in $\partial(\Delta')$,

$$J(\Delta')_\gamma = J(\Delta)_\gamma$$

examples of simple and split subdivisions

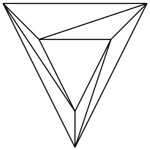

Figure: Δ

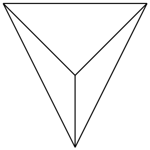

Figure: Δ''

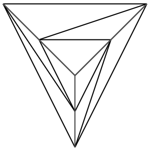

Figure: Δ'

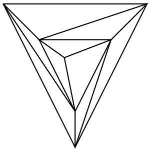

Figure: $\widetilde{\Delta}'$

main result

Theorem. If Δ' is a split subdivision of Δ and both $S^r(\widehat{\Delta})$ and $S^r(\widehat{\Delta}'')$ are free, then

$$S^r(\widehat{\Delta}') \simeq S^r(\widehat{\Delta}) \bigoplus \left(S^r(\widehat{\Delta}'')/\mathbb{R}[x_0, \ldots, x_k]\right),$$

and $S^r(\widehat{\Delta}')$ is free.

starting point, Schenck, 2014

Theorem. Let $A(T_k)$ be the Alfeld split of an k-simplex T_k in $\mathbb{R}^k$. Then

$$\dim S_d^r(A(T_k)) = \binom{d+k}{k} + A(k,d,r),$$

where

$$A(k,d,r) := \begin{cases} k\binom{d+k-\frac{(r+1)(k+1)}{2}}{k}, & \text{if r is odd,} \\ \binom{d+k-1-\frac{r(k+1)}{2}}{k} + \cdots + \binom{d-\frac{r(k+1)}{2}}{k}, & \text{if r is even.} \end{cases}$$

Moreover, the associated module of splines $S^r(\widehat{A}(T_k))$ is free for any r.

• Alfeld split $A(T_k)$ of an k-dimensional simplex T_k in $\mathbb{R}^k$, is obtained from a single simplex T_k by adding a single interior vertex u, and then coning over the boundary of T_k.

subdivisions: facet split

For a full-dimensional k-simplex $T_k := [v_0, v_1, \dots, v_k] \subseteq \mathbb{R}^k$, start with the Alfeld split $A(T_k)$ with the interior vertex u.

For each $i = 0, \dots, k$, let F_i be the facet of T_k opposite vertex v_i. Let u_i be the point strictly interior to F_i and collinear with v_i and u. Each u_i induces a $(k-1)$-dimensional Alfeld split $A(F_i)$ of F_i. Cone u over $A(F_i)$ forming a pyramid P_i in $\mathbb{R}^k$.

The collection of $k+1$ pyramids P_i is the facet split $F(T_k)$.

subdivisions: double Alfeld split

For a full-dimensional k-simplex $T_k := [v_0, v_1, \ldots, v_k] \subseteq \mathbb{R}^k$, start with the Alfeld split $A(T_k)$ with the interior vertex u.

For each $i = 0, \ldots, k$, let F_i be the facet of T_k opposite vertex v_i. Let u_i be a point strictly interior to the simplex $T_k^i := [u, F_i]$ and collinear with v_i and u. Each u_i induces an Alfeld split $A(T_k^i)$ of T_k^i.

The collection of $k+1$ Alfeld splits $A(T_k^i)$ is the double Alfeld split $AA(T_k)$.

2D subdivisions

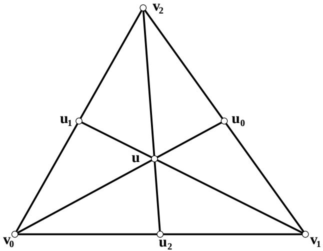

Figure: $F(T_2)$

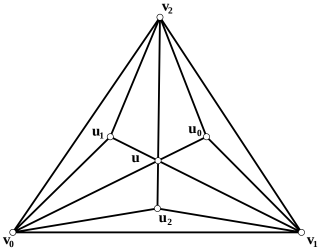

Figure: $AA(T_2)$

3D subdivisions

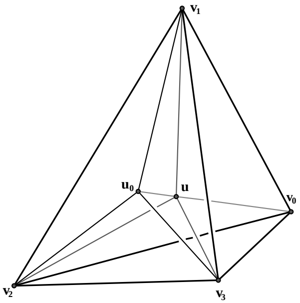

Figure: A part of $F(T_3)$

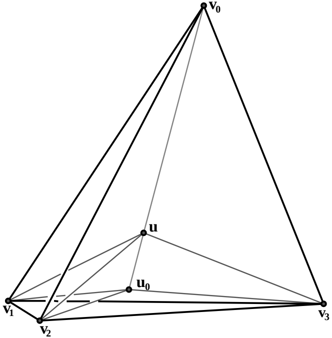

Figure: A part of $AA(T_3)$

main result

Let $F(T_k)$ and $AA(T_k)$ be the facet and double Alfeld splits. Then

$$\dim S_d^r(F(T_k)) = \binom{d+k}{k} + A(k,d,r) + (k+1)P(k,d,r),$$

$$\dim S_d^r(AA(T_k)) = \binom{d+k}{k} + (k+2)A(k,d,r),$$

$$A(k,d,r) := \begin{cases} k\binom{d+k-\frac{(r+1)(k+1)}{2}}{k}, & \text{if r is odd,} \\ \binom{d+k-1-\frac{r(k+1)}{2}}{k} + \cdots + \binom{d-\frac{r(k+1)}{2}}{k}, & \text{if r is even.} \end{cases}$$

$$P(k,d,r) := \begin{cases} (k-1)\binom{d+k-\frac{(r+1)k}{2}}{k}, & \text{if r is odd} \\ \binom{d+k-1-\frac{rk}{2}}{k} + \cdots + \binom{d+1-\frac{rk}{2}}{k}, & \text{if r is even.} \end{cases}$$

remarks

- The proof of the main result holds for partial facet and double Alfeld splits, i.e. for the case where not every tetrahedron in $A(T_k)$ is subdivided. Such partial subdivisions are useful in the context of boundary finite elements.

- The requirement of the collinearity in the definitions for the facet and double Alfeld splits can be omitted for $r = 1$.

- Computations in the Macaulay2 package of Grayson and Stillman (available at `http://www.math.uiuc.edu/Macaulay2`) and in Alfeld's spline software (available at `http://www.math.utah.edu/~pa`) were essential to this work.

- We also thank the Mathematische Forschungsinstitut Oberwolfach, where our collaboration began.

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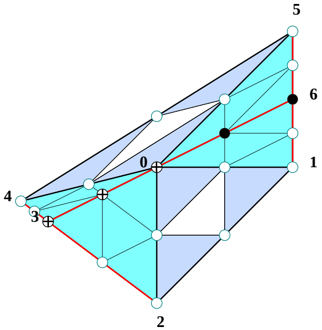
starting point

T. S., Redundancy of smoothness conditions and supersmoothness of bivariate splines. IMA Journal of Numerical Analysis, Vol. 34, Number 3, 2014, 1701–1714

Lemma. Let Δ be a cell with four non-collinear edges meeting at the point u. Then there exists a unique straight line passing through u with the property that for any smooth quadratic spline s on Δ, the restriction of s on this line is a univariate quadratic polynomial.

example



number of vertices 7, number of triangles 6; coordinates of the vertices:

$$(0,0),\ (200,0),\ (0,200),\ (-160,80),\ (-200,50),\ (200,-200),\ (200,-100),$$

and connectivities: $(0\ 1\ 2),\ (0\ 2\ 3),\ (0\ 3\ 4),\ (0\ 4\ 5),\ (0\ 5\ 6),\ (0\ 6\ 1).$

Set $r = 1$, $d = 2$, and supersmoothness two across the edges $[v_0, v_3]$, and $[v_0, v_6]$. This makes the partition into a cell with four interior non-collinear edges. The line $[v_3, v_6]$ is the line l from the Lemma.

comments and questions

- if Δ be a cell with four edges, and three slopes, i.e., two edges are collinear, then the straight line from the Lemma is the one formed by the collinear edges

- the result above can be easily generalized to smoothness r degree $r + 1$; probably the lemma can be too

- both results can be restated in terms of supersmoothness: i.e. the second derivatives match in certain directions

- what about a different number of non-collinear edges in a cell?

- what is the geometric significance of this line?