Basis and dimension of trivariate geometrically continuous isogeometric functions on two-patch domains

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Content

Isogeometric Analysis (IGA)

T.J.R.Hughes, J.A.Cottrell and Y.Bazilevs.
Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement.
Computer methods in applied mechanics and engineering, 194(39), 4135–4195, 2005.

- Approximation method for PDEs,
- alternative to standard FEM.
- Use same basis functions to represent the geometry and to approximate the solutions of PDEs.

Advantages:

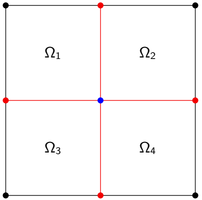
- smooth basis functions with compact support,
- perform computations on exact geometry.

Isogeometric Analysis (IGA)

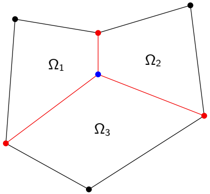
IGA allows discretization spaces of high order smoothness.

- Need multi-patch parameterization.

Question:
How to obtain smooth (C^1) functions on multi-patch domains?



Trivial



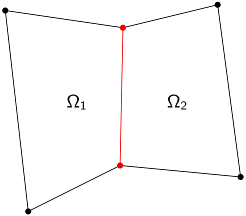
Non-trivial

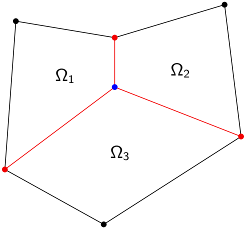
Related work - planar domains

C^1-smooth isogeometric function spaces on *bilinearly* parametrized planar domains:

- M.Kapl, V.Vitrih, B.Jüttler and K.Birner.
  Isogeometric analysis with geometrically continuous functions on two-patch geometries.
  Computers & Mathematics with Applications, 70(7), 1518–1538, 2015.

- M.Kapl, F.Buchegger, M.Bercovier and B.Jüttler.
  Isogeometric analysis with geometrically continuous functions on planar multi-patch geometries.
  Computer Methods in Applied Mechanics and Engineering, 316, 209–234, 2017.



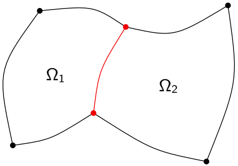


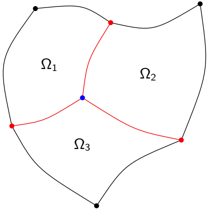
Related work - planar domains

C^1-smooth isogeometric function spaces on *more general* parametrized planar domains:

M.Kapl, G.Sangalli and T.Takacs.
Dimension and basis construction for analysis-suitable G^1 two-patch parameterizations.
Computer Aided Geometric Design, 52, 75–89, 2017.

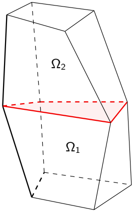
M.Kapl, G.Sangalli and T.Takacs.
Construction of analysis-suitable G^1 planar multi-patch parameterizations.
arXiv preprint arXiv:1706.03264, 2017.

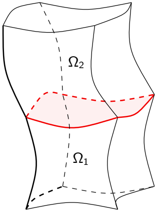




Extend these results to volumetric domains

Given: Hexahedral volumetric two-patch domain $\Omega = \Omega_1 \cup \Omega_2$.





Goal: Dimension and basis of C^1-smooth isogeometric functions on Ω.

Future work: 4^{th}-order PDEs, e.g biharmonic equation.

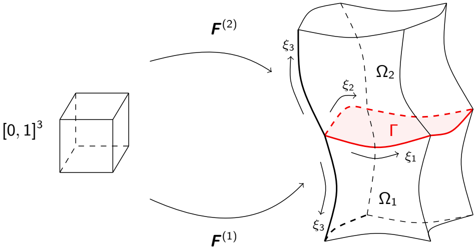
Volumetric setting

- Common face parameterized by $\Gamma = [0,1]^2 \times \{0\}$.
- Parametric representations $\boldsymbol{F}^{(1)}, \boldsymbol{F}^{(2)}$ with coordinate functions from

$$\mathcal{P} = \mathcal{S}_k^p \otimes \mathcal{S}_k^p \otimes \mathcal{S}_k^p,$$

- $\mathcal{S}_k^p$ space of spline functions on $[0,1]$ of degree p with k uniformly distributed inner knots of multiplicity $p-1$,
- two-patch geometry mapping $\boldsymbol{F} = (\boldsymbol{F}^{(1)}, \boldsymbol{F}^{(2)}) \in C^0(\Omega)$ where

$$\boldsymbol{F}^{(1)} = \boldsymbol{F}^{(2)} \quad \text{on } \Gamma.$$



Space of C^1-smooth isogeometric functions

Isogeometric function $\nu \in (\mathcal{P} \times \mathcal{P}) \circ \boldsymbol{F}^{-1}$

$$(\nu|_{\Omega^{(i)}})(\boldsymbol{x}) = \nu^{(i)}(\boldsymbol{x}) = (w^{(i)} \circ (\boldsymbol{F}^{(i)})^{-1})(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega^{(i)}$$

with $w^{(1)}, w^{(2)} \in \mathcal{P}$.

Space of C^1-smooth isogeometric functions $\mathcal{V}_{\boldsymbol{F}}$

$$\mathcal{V}_{\boldsymbol{F}} = \left[(\mathcal{P} \times \mathcal{P}) \circ \boldsymbol{F}^{-1}\right] \cap C^1(\Omega_1 \cup \Omega_2).$$

G^1-continuity condition

Associated graph surface $\mathbf{\Phi} = (\mathbf{\Phi}^{(1)}, \mathbf{\Phi}^{(2)})$, $\mathbf{\Phi}^{(i)} = (\boldsymbol{F}^{(i)}, w^{(i)}) \in \mathbb{R}^4$.

- Isogeometric function ν is C^1-smooth if $\mathbf{\Phi}$ is G^1-smooth (G^1-smooth $=$ C^1-smooth after reparameterization).

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$\mathbf{\Phi}$ is G^1-smooth iff the two patches have identical tangent hyperplanes along the common interface

$$\mathbf{\Phi}^{(1)} = \mathbf{\Phi}^{(2)} \quad \text{on } \Gamma.$$

G^1-continuity condition

Identical tangent hyperplanes:

- $w^{(1)}(\xi_1, \xi_2, 0) = w^{(2)}(\xi_1, \xi_2, 0)$.
- $\partial_1 \mathbf{\Phi}^{(1)}(\xi_1, \xi_2, 0) = \partial_1 \mathbf{\Phi}^{(2)}(\xi_1, \xi_2, 0)$,
  $\partial_2 \mathbf{\Phi}^{(1)}(\xi_1, \xi_2, 0) = \partial_2 \mathbf{\Phi}^{(2)}(\xi_1, \xi_2, 0)$,
  $\partial_3 \mathbf{\Phi}^{(1)}(\xi_1, \xi_2, 0)$ and $\partial_3 \mathbf{\Phi}^{(2)}(\xi_1, \xi_2, 0)$
  are linearly dependent at each point of the interface.

$$\longrightarrow \qquad M(\xi_1, \xi_2) = \det \begin{pmatrix} \nabla \boldsymbol{F}^{(1)}|_\Gamma & \partial_3 \boldsymbol{F}^{(2)}|_\Gamma \\ \nabla w^{(1)}|_\Gamma & \partial_3 w^{(2)}|_\Gamma \end{pmatrix} = 0$$

G^1-CONTINUITY CONDITION

$$M(\xi_1, \xi_2) = \det \begin{pmatrix} \nabla \boldsymbol{F}^{(1)}|_\Gamma & \partial_3 \boldsymbol{F}^{(2)}|_\Gamma \\ \nabla w^{(1)}|_\Gamma & \partial_3 w^{(2)}|_\Gamma \end{pmatrix} =$$

$$\alpha_1\, \partial_1 w^{(1)}|_\Gamma - \alpha_2\, \partial_2 w^{(1)}|_\Gamma + \alpha_3\, \partial_3 w^{(1)}|_\Gamma - \alpha_4\, \partial_3 w^{(2)}|_\Gamma = 0, \qquad (*)$$

where

$$\alpha_1 = \det(\partial_2 \boldsymbol{F}^{(1=2)}|_\Gamma\, \partial_3 \boldsymbol{F}^{(1=2)}|_\Gamma\, \partial_3 \boldsymbol{F}^{(2)}|_\Gamma)$$

$$\alpha_2 = \det(\partial_1 \boldsymbol{F}^{(1=2)}|_\Gamma\, \partial_3 \boldsymbol{F}^{(1)}|_\Gamma\, \partial_3 \boldsymbol{F}^{(2)}|_\Gamma)$$

$$\alpha_3 = \det(\partial_1 \boldsymbol{F}^{(1=2)}|_\Gamma\, \partial_2 \boldsymbol{F}^{(1=2)}|_\Gamma\, \partial_3 \boldsymbol{F}^{(2)}|_\Gamma) = \det(\nabla \boldsymbol{F}^{(2)}|_\Gamma)$$

$$\alpha_4 = \det(\partial_1 \boldsymbol{F}^{(1=2)}|_\Gamma\, \partial_2 \boldsymbol{F}^{(1=2)}|_\Gamma\, \partial_3 \boldsymbol{F}^{(1)}|_\Gamma) = \det(\nabla \boldsymbol{F}^{(1)}|_\Gamma)$$

Gluing data

$$\alpha_1 \, \partial_1 w^{(1)}|_{\Gamma} - \alpha_2 \, \partial_2 w^{(1)}|_{\Gamma} + \alpha_3 \, \partial_3 w^{(1)}|_{\Gamma} - \alpha_4 \, \partial_3 w^{(2)}|_{\Gamma} = 0 \qquad (*)$$

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ called *gluing data*, is defined by the first three coordinates of $\Phi^{(i)}$, i.e. $\boldsymbol{F}^{(i)}$.

For a given geometry mapping $\boldsymbol{F} = (\boldsymbol{F}^{(1)}, \boldsymbol{F}^{(2)})$ the gluing data can be computed from $\boldsymbol{F}$.

- We call gluing data derived from known geometry mapping $\boldsymbol{F}$ *geometric gluing data* $D_{\boldsymbol{F}}$.
- If gluing data $D_{\boldsymbol{F}}$ is derived from *trilinear* $\boldsymbol{F} = (\boldsymbol{F}^{(1)}, \boldsymbol{F}^{(2)})$ we call it *trilinear geometric gluing data*.

Glued spline space

- General *gluing data*

$$D = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \Pi^{\boldsymbol{q}_1} \times \Pi^{\boldsymbol{q}_2} \times \Pi^{\boldsymbol{q}_3} \times \Pi^{\boldsymbol{q}_4},$$

  where $\Pi^{\boldsymbol{q}}$ denotes the space of bivariate tensor-product polynomials of bi-degree $\boldsymbol{q}$, with the four bi-degrees $\boldsymbol{Q} = [\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4]$.

  Regular gluing data:

$$\alpha_3(s,t)\ \alpha_4(s,t) \neq 0 \quad \forall (s,t) \in [0,1]^2 .$$

- *Glued spline space:* $\mathcal{G}_D \subseteq \mathcal{P} \times \mathcal{P}$ with

$$\mathcal{G}_D = \Big\{ \boldsymbol{f} = (f^{(1)}, f^{(2)}) \in \mathcal{P}^2 \ : \ \underbrace{f^{(1)} = f^{(2)} \text{ on } \Gamma}_{(**)} \text{ and}$$

$$\underbrace{\alpha_1 \partial_1 f^{(1)} - \alpha_2 \partial_2 f^{(1)} + \alpha_3 \partial_3 f^{(1)} - \alpha_4 \partial_3 f^{(2)} = 0 \text{ on } \Gamma}_{(*)} \Big\}.$$

Space $\mathcal{V}_{\boldsymbol{F}}$

Space of C^1-smooth isogeometric functions $\mathcal{V}_{\boldsymbol{F}}$

$$\mathcal{V}_{\boldsymbol{F}} = \left[(\mathcal{P} \times \mathcal{P}) \circ \boldsymbol{F}^{-1}\right] \cap C^1(\Omega_1 \cup \Omega_2).$$

Theorem

Consider regular gluing data D and regular geometry mapping $\boldsymbol{F} \in \mathcal{G}_D^3$. Any C^1–smooth isogeometric function is the push-forward of a glued spline function,

$$\mathcal{V}_{\boldsymbol{F}} = \mathcal{G}_D \circ \boldsymbol{F}^{-1}.$$

Closely related to:

D.Groisser and J.Peters.
Matched G^k-constructions always yield C^k-continuous isogeometric elements.
Computer Aided Geometric Design, 34, 67–72, 2015.

COMPUTATION OF THE DIMENSION OF $\mathcal{G}_D$

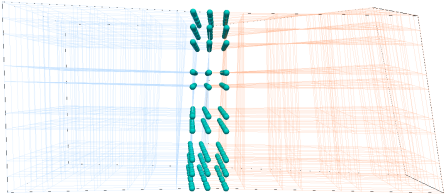
- To avoid numerical errors we use rational numbers.
- We compute various instances to obtain a lower bound of $\dim \mathcal{G}_D$
- and use random numbers to get the *generic*[1] dimension.
- $\dim \mathcal{G}_D = \dim(\ker A_D)$.

Interpolation gives closed formulas for the generic dimension of $\mathcal{G}_D$.

[1] generic: valid with probabilty 1.

Types of basis functions

We distinguish between two types of basis functions:



- *Inner basis functions* (B-splines)

$$2(p+1+k(p-1))^2(p-1+k(p-1)), \tag{1}$$

- *interface basis functions*.

Trilinear geometric gluing data

Remember: $\mathcal{S}_k^p$ with spline degree p and k uniformly distributed inner knots of multiplicity $p-1$.

k	p=2	p=3	p=4	p=5	p=6
0	18+10	64+20	150+34	288+52	490+74
1	64+10	288+29	768+65	1600+117	2880+185
2	150+10	768+40	2178+106	4704+208	8670+346
3	288+10	1600+53	4704+157	10368+325	19360+557
4	490+10	2880+68	8670+218	19360+468	36450+818

- Interface basis functions, valid for $p \geq 3$:

$$2 + 2k + 13k^2 - 10p\,k(1+k) + 2p^2\,(k+1)^2. \qquad (2)$$

The dimension of the glued spline space is given as (1) + (2).

Basis computation

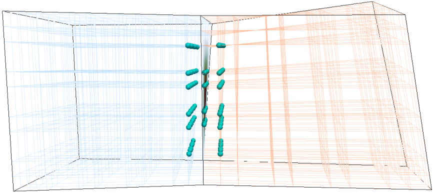
- Interface basis functions.
- Locally supported functions,
  - C^1 boundary conditions.

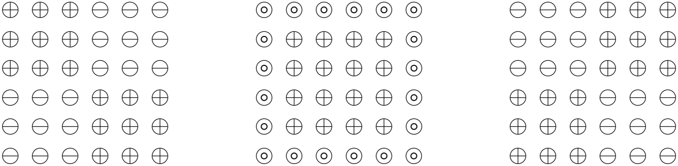
Local support: size of the support is independent of the number k of inner knots.

Basis trilinear geometric gluing data

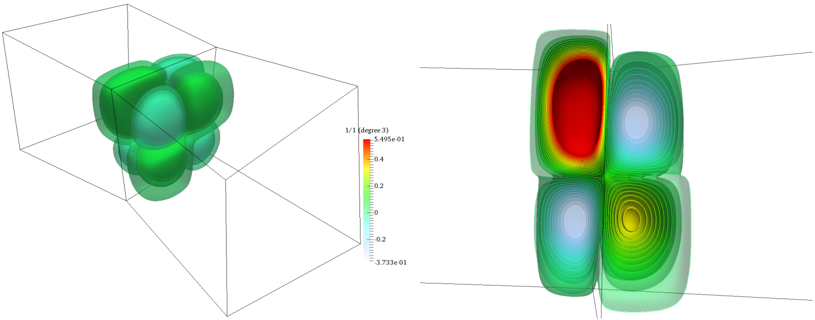
$p = 3$

- $\exists$ locally supported basis functions if $k > 2$,
- one type of functions and $(k-2)^2$ (scaled) translates.





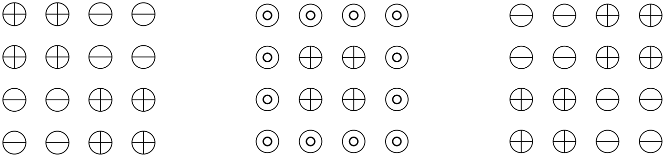
Basis function for $p = 3$



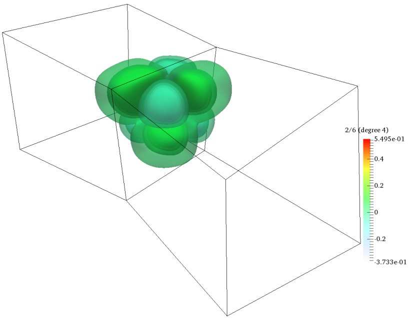
Basis trilinear geometric gluing data

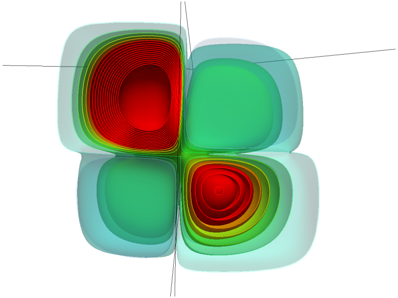
$p = 4$

- $\exists$ locally supported basis functions if $k > 1$,
- six types of functions, in total $(5k^2 - 6k + 2)$



Basis function for $p = 4$





Conclusion and future work

- Planar results extended to volumetric case,
- definition of gluing data and glued spline space,
- space of C^1-smooth isogeometric functions,
- results on the dimension and a basis for the glued spline space.

- Further investigation of basis functions for other types of gluing data,
- approximation power,
- solving 4^{th}-order PDEs.