Commutative Algebra in Approximation Theory

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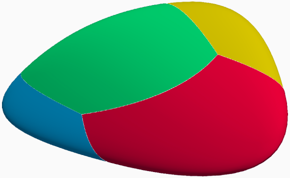
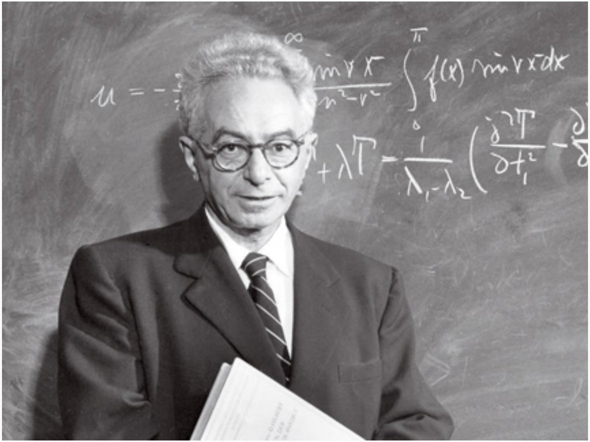


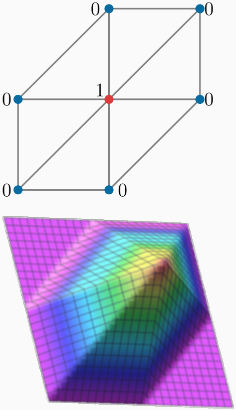
Workshop on the Applications of Commutative Algebra

The Fields Institute, May 26, 2025

Courant hat function



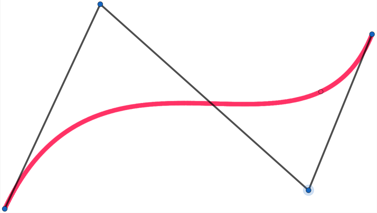
Courant (1888-1972)

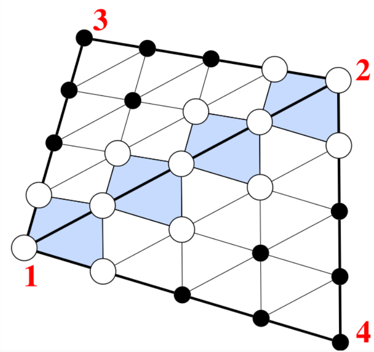


Bézier curves

Bézier (1910-1999)

De Casteljau (1930-2022)





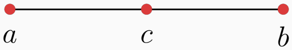
Bernstein–Bézier methods

Spline functions (definition)

Splines are piecewise polynomial functions with a specified order of smoothness on polyhedral partitions in $\mathbb{R}^n$.

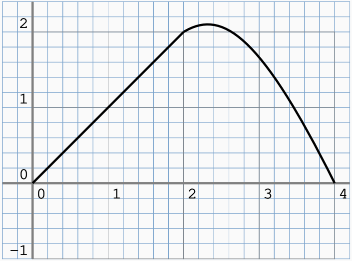
Ex: A spline function on $[a, c] \cup [c, b]$ is any function of the form

$$f(x) = \begin{cases} f_1(x) & \text{if} \quad x \in [a, c] \\ f_2(x) & \text{if} \quad x \in [c, b]; \end{cases} \quad \text{for } f_1(x), f_2(x) \in \mathbb{R}[x].$$



Taking $a = 0$, $c = 2$ and $b = 4$ then

$$f(x) = \begin{cases} x & \text{if} \ \ x \in [0, 2] \\ -x^2 + 5x - 4 & \text{if} \ \ x \in [2, 4] \end{cases}$$



Univariate splines

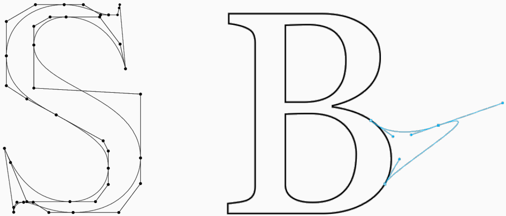
Prop: The spline $f = (f_1, f_2)$ defines a C^r-continuous function on $[a, b]$ $\Leftrightarrow$ the polynomial $f_1 - f_2$ is divisible by $(x - c)^{r+1}$:

$$f_1 - f_2 \in \langle (x - c)^{r+1} \rangle \subseteq \mathbb{R}[x].$$

- The C^r-continuous splines $\mathcal{S}^r([a, b])$ is a vector subspace of $\mathbb{R}[x]^2$.
- If $\deg f_i \leq d$ then $\mathcal{S}^r_d([a, b])$ is a finite-dimensional vector space.

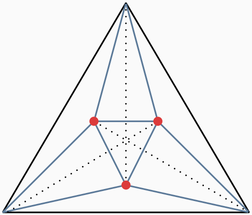
For any $(f_1, f_2) \in \mathcal{S}^r([a, b])$, we can write $(f_1, f_2) = (f_1, f_1) + (0, f_2 - f_1)$.

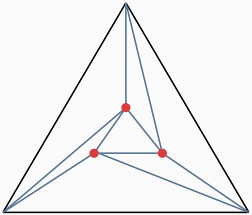
- Basis: $\{(1, 1), (x, x), \ldots, (x^d, x^d), (0, (x - c)^{r+1}), \ldots, (0, (x - c)^d)\}$.



Dimension of splines on triangulations

- **Strang's conjecture** (1974): the dimension of a spline space over a triangulation is given by a combinatorial formula.
- Morgan and Scott (1975) proved a dimension formula for polynomial degree $d \geq 5$ and smoothness $r = 1$. Strang's conjecture is not valid for arbitrary triangulations.





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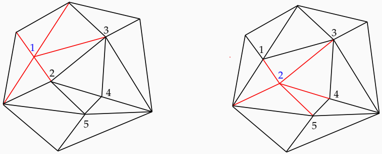
- **Schumaker** (1984) proved *combinatorial* lower and upper bounds for splines on arbitrary triangulations

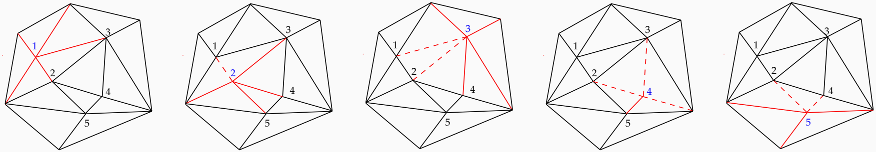


$t_1 = 3$ $t_2 = 5$

For this numbering, the lower bound formula gives $\dim \mathcal{S}_2^1(\Delta) \geq 9$.

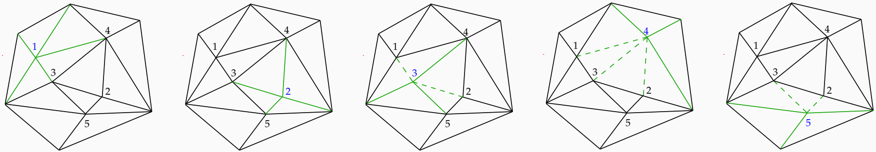
Bounds on the dimension of splines on triangulations

We have $f_2^0 = 14$ triangles, $f_1^0 = 18$ edges, and $f_0^0 = 5$ vertices.



$\tilde{t}_1 = 3$ $\tilde{t}_2 = 4$ $\tilde{t}_3 = 4$ $\tilde{t}_4 = 1$ $\tilde{t}_5 = 3$

The upper bound for this numbering gives $\dim \mathcal{S}_2^1(\Delta) \leq 11$.



$\tilde{t}_1 = 3$ $\tilde{t}_2 = 3$ $\tilde{t}_2 = 3$ $\tilde{t}_3 = 3$ $\tilde{t}_5 = 3$

For this numbering we get $\dim \mathcal{S}_2^1(\Delta) \leq 9$, which implies $\dim \mathcal{S}_2^1(\Delta) = 9$.

Dimension of splines on triangulations: state of the art

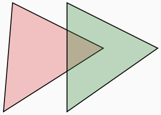
- **Strang's conjecture** (1974): combinatorial formula.
- Morgan and Scott (1975) proved a dimension formula for polynomial degree $d \geq 5$: Strang's conjecture is not valid for arbitrary triangulations.
- **Schumaker** (1984) proved combinatorial lower and upper bounds for splines on arbitrary triangulations
- Alfeld (1987) proved dimension formula for $d \geq 4r+1$. The results were extended to $d \geq 3r+2$ by Hong (1991).
- **Billera** (1988) introduced the use of **homological algebra** in the study of splines and poved Strang's conjecture for generic triangulations $\mathcal{S}_d^1(\Delta)$.
- Stillman and Yuan (2019) counter-example to the **Schenck–Stiller** $2r+1$ conjecture. The conjecture is still open for $\boldsymbol{\mathcal{S}_3^1(\Delta)}$.
- Schenck, Stillman, and Yuan (2020): combinatorial formula does not hold in general for $d \leq \frac{22r+7}{10}$.

Multivariate splines

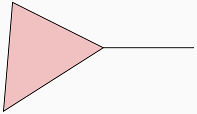
Splines $\mathcal{S}^r(\Delta)$ are piecewise polynomial functions of smoothness r on a given polyhedral complex Δ embedded in $\mathbb{R}^n$.

A **polyhedral complex** $\Delta \subset \mathbb{R}^n$ is a finite collection of polytopes such that

- the faces of each polytope in Δ is also in Δ,
- the intersection of any two polytopes in Δ is also in Δ.



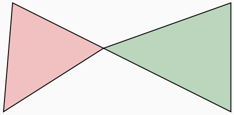
Pure: Every maximal element of Δ is an n-dimensional polyhedron.



Hereditary: For any n-dimensional faces $\sigma, \sigma' \in \Delta_n$ such that $\tau \in \sigma \cap \sigma'$ there are

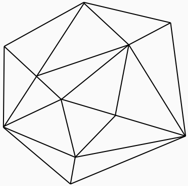
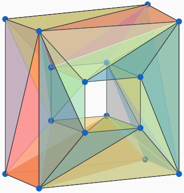
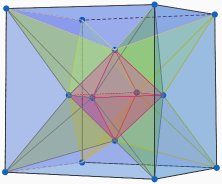
$$\sigma = \sigma_1,\ \sigma_2,\ \ldots,\ \sigma_m = \sigma' \in \Delta_n$$

s.t. $\tau \in \sigma_i$, and σ_i and σ_{i+1} are adjacent.



Multivariate splines

Let $\Delta \subset \mathbb{R}^n$ be a pure, hereditary n-dimensional polyhedral complex, and $r, d \geq 0$ be integers.

The space of **splines** on Δ is defined as

$$\boxed{\mathcal{S}^r(\Delta)} = \left\{ f \in C^r(\Delta) \colon f|_\sigma \in \mathbb{R}[x_1, \ldots, x_n] \text{ for all } \sigma \in \Delta_n \right\}$$

$$\boxed{\mathcal{S}^r_d(\Delta)} = \left\{ f \in \mathcal{S}^r(\Delta) \colon \deg(f|_\sigma) \leq d \text{ for all } \sigma \in \Delta_n \right\}.$$

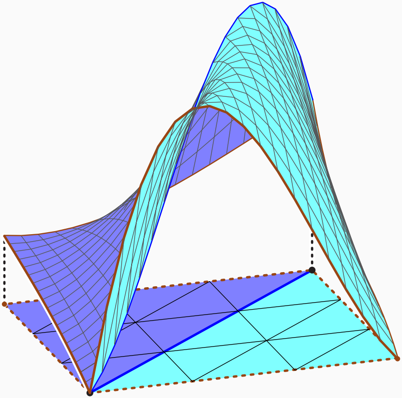
- The set $\mathcal{S}^r_d(\Delta)$ is a real vector space.

Smoothness condition

Algebraically, if $\sigma, \sigma' \in \Delta_n$ and $\sigma \cap \sigma' = \tau \in \Delta_{n-1}$ then

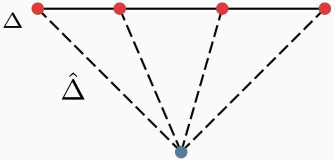
$$f \in \mathcal{S}^r(\Delta) \quad \Leftrightarrow \quad f|_\sigma - f|_{\sigma'} \in \left\langle \ell_\tau^{r+1} \right\rangle$$

where ℓ_τ is a linear polynomial vanishing on τ.



Splines as graded modules

If we embed Δ in $\{x_{n+1} = 1\} \subseteq \mathbb{R}^{n+1}$, we can consider the splines $\mathcal{S}^r(\hat{\Delta})$ on the new polyhedral complex $\hat{\Delta}$.



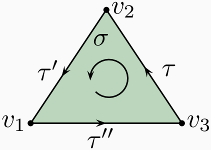
- Given $f = (f_1, \ldots, f_m) \in \mathcal{S}^r_d(\Delta)$, the homogenization

$$f^h = (f_1^h, \ldots, f_m^h) \in \mathcal{S}^r_d(\hat{\Delta}).$$

- $\mathcal{S}^r(\hat{\Delta}) = \bigoplus_{d \geq 0} \mathcal{S}^r(\hat{\Delta})_d$ is a graded module, and $\dim \mathcal{S}^r_d(\Delta) = \dim \mathcal{S}^r(\hat{\Delta})_d$
- Then, to study the dimension of $\mathcal{S}^r_d(\Delta)$, it suffices to study the **Hilbert series** of the module $\mathcal{S}^r(\hat{\Delta})$.

Simplicial complex

Let Δ be a **simplicial complex**, to a simplex associate an **orientation**:



$$[\sigma] = [v_1, v_2, v_3]$$
$$[\tau] = [v_2, v_3]$$
$$[\tau'] = [v_1, v_2]$$

Given Δ and a ring R, the R-module C_i is generated by the oriented i-simplices:

$$[v_{j_0}, \ldots, v_{j_i}] = (-1)^{\mathrm{sgn}(\rho)}[v_{j\rho(0)}, \ldots, v_{j\rho(i)}] \qquad \text{permutation } \rho$$

A simplicial complex gives rise to a **chain complex**: the boundary map

$$\partial([\sigma]) = \sum_{j=0}^{n}(-1)^j[v_{i_0}, \ldots, \hat{v_{ij}}, \ldots, v_{i_n}] \qquad \partial\partial = 0$$

Ex: $\partial([v_1, v_2, v_3]) = [v_2, v_3] - [v_1, v_3] + [v_1, v_2]$

We extend ∂ by linearity $C_i \xrightarrow{\partial_i} C_{i-1}$ $\qquad \mathrm{Im}(\partial_{i+1}) \subseteq \ker(\partial_i)$

$$\mathcal{C}: \quad 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \to 0$$

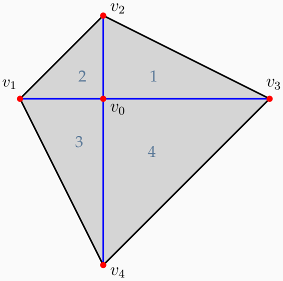
Homology modules: $\quad H_i(\mathcal{C}) = \ker(\partial_i)/\,\mathrm{Im}(\partial_{i+1})$.

Relative homology with respect the boundary

Given a subspace A of X we can consider $C_i(X,A) = C_i(X)/C_i(A)$:

$$0 \to C_n(X,A) \xrightarrow{\partial_n} C_{n-1}(X,A) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0(X,A) \to 0$$

For a simplicial complex Δ, we consider the relative homology of Δ with respect to the boundary $\partial(\Delta)$, and the ring $R = \mathbb{R}[x_0, \ldots, x_n]$.



Ex: $0 \to \mathbb{R}[x,y]^4 \to \mathbb{R}[x,y]^4 \to \mathbb{R}[x,y] \to 0$

In general, to a simplicial complex Δ associate

$$\mathcal{R}\colon \quad 0 \to \mathcal{R}_n \to \mathcal{R}_{n-1} \to \cdots \to \mathcal{R}_0 \to 0$$

where $\mathcal{R}_i = \bigoplus_{\delta \in \Delta_i^0} \mathcal{R}(\delta)$, with $\mathcal{R}(\delta) = R$.

The set of interior i-dimensional faces of Δ is Δ_i^0.

Dimension formula for the spline space

For $R = \mathbb{R}[x_0, \ldots, x_n]$ and $r \geq 0$, define the **complex of ideals** $\mathcal{J}$ on the simplicial complex Δ:

$$\begin{aligned} \mathcal{J}(\sigma) &= \langle 0 \rangle && \text{for } \sigma \in \Delta_n \\ \mathcal{J}(\tau) &= \langle \ell_\tau^{r+1} \rangle && \text{for } \tau \in \Delta_{n-1}^0 \\ &\vdots \\ \mathcal{J}(\gamma) &= \sum_{\tau \ni \gamma} \langle \ell_\tau^{r+1} \rangle && \text{for } \gamma \in \Delta_0^0 \end{aligned}$$

Define the quotient complex by $\mathcal{R}/\mathcal{J}(\delta) = \mathcal{R}(\delta)/\mathcal{J}(\delta)$:

$$\mathcal{R}/\mathcal{J}\colon\ 0 \longrightarrow \bigoplus_{\sigma \in \Delta_n} \mathcal{R}(\sigma) \xrightarrow{\partial_n} \bigoplus_{\tau \in \Delta_{n-1}^0} \mathcal{R}/\mathcal{J}(\tau) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} \bigoplus_{\gamma \in \Delta_0^0} \mathcal{R}/\mathcal{J}(\gamma) \longrightarrow 0$$

We have: $\boxed{\dim \mathcal{S}_d^r(\Delta) = \dim H_n(\mathcal{R}/\mathcal{J})_d}$.

Dimension formula for the spline space

The short exact sequence of complexes $0 \to \mathcal{J} \to \mathcal{R} \to \mathcal{R}/\mathcal{J} \to 0$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \bigoplus_{\tau\in\Delta^0_{n-1}} \mathcal{J}(\tau) \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus_{\gamma\in\Delta^0_0} \mathcal{J}(\gamma) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{\sigma\in\Delta_n} \mathcal{R}(\sigma) \arrow[r] \arrow[d] & \bigoplus_{\tau\in\Delta^0_{n-1}} \mathcal{R}(\tau) \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus_{\gamma\in\Delta^0_0} \mathcal{R}(\gamma) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{\sigma\in\Delta_n} \mathcal{R}(\sigma) \arrow[r] \arrow[d] & \bigoplus_{\tau\in\Delta^0_{n-1}} \mathcal{R}/\mathcal{J}(\tau) \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus_{\gamma\in\Delta^0_0} \mathcal{R}(\gamma) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & & 0 &
\end{tikzcd}
$$

gives rise to a long exact sequence of homology modules:

$$\cdots \to H_{i+1}(\mathcal{R}/\mathcal{J}) \to H_i(\mathcal{J}) \to H_i(\mathcal{R}) \to H_i(\mathcal{R}/\mathcal{J}) \to H_{i-1}(\mathcal{J}) \to \cdots$$

For Δ a pure and hereditary simplicial complex:

- $H_i(\mathcal{R}) = 0$ except for $H_n(\mathcal{R}) = R$,
- $H_0(\mathcal{R}/\mathcal{J}) = 0$,
- $H_i(\mathcal{R}/\mathcal{J}) \cong H_{i-1}(\mathcal{J})$ for $i \leq n-1$,
- $\mathcal{S}^r(\hat{\Delta}) \cong R \oplus H_1(\mathcal{J})$.

Dimension of spline spaces on triangulations

$$\dim \mathcal{S}_d^r(\Delta) = \sum_{\sigma\in\Delta_2} \dim \mathcal{R}(\sigma)_d - \sum_{\tau\in\Delta_1^0} \dim \mathcal{R}/\mathcal{J}(\tau)_d + \sum_{\gamma\in\Delta_0^0} \dim \mathcal{R}/\mathcal{J}(\gamma)_d + \dim H_0(\mathcal{J})_d$$

We know:

$$\sum_{\sigma\in\Delta_2} \dim \mathcal{R}(\sigma)_d = f_2 \binom{d+2}{2}$$

(f_i^0 is the number interior i-faces)

$$\sum_{\tau\in\Delta_1^0} \dim \mathcal{R}/\mathcal{J}(\tau)_d = f_1^0 \left[\binom{d+2}{2} - \binom{d+2-(r+1)}{2}\right]$$

For computing $\dim \mathcal{R}/\mathcal{J}(\gamma)_d$ we use the resolution

$$0 \to \mathcal{R}(-\Omega_i - 1)^{a_i} \oplus \mathcal{R}(-\Omega_i)^{b_i} \to \oplus_{j=1}^{t_i} \mathcal{R}(-r-1) \to \mathcal{R} \to \mathcal{R}/\mathcal{J}(\gamma_i) \to 0$$

where t_i as the number of different slopes of the edges containing γ_i and

$$\Omega = \left\lfloor \frac{t\,r}{t-1} \right\rfloor + 1, \quad a = t\,(r+1) + (1-t)\,\Omega, \quad b = t - 1 - a.$$

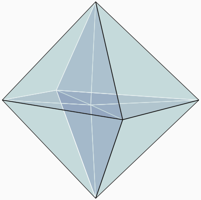
Dimension of trivariate spline spaces

Given a 3-dim simplicial complex Δ, take $R = \mathbb{R}[x, y, z]$:

$$0 \to \bigoplus_{\iota \in \Delta_3} \mathcal{R}(\iota) \xrightarrow{\partial_3} \bigoplus_{\sigma \in \Delta_2^0} \mathcal{R}/\mathcal{J}(\sigma) \xrightarrow{\partial_2} \bigoplus_{\tau \in \Delta_1^0} \mathcal{R}/\mathcal{J}(\tau) \xrightarrow{\partial_1} \bigoplus_{\gamma \in \Delta_0^0} \mathcal{R}/\mathcal{J}(\gamma) \to 0$$

- Then $\dim \mathcal{S}_d^r(\Delta) = \sum_{\iota \in \Delta_3} \dim \mathcal{R}(\iota)_d - \sum_{\sigma \in \Delta_2^0} \dim \mathcal{R}/\mathcal{J}(\sigma)_d + \sum_{\tau \in \Delta_1^0} \dim \mathcal{R}/\mathcal{J}(\tau)_d$

$$- \boxed{\dim R/J(\gamma)_d} + \dim H_1(\mathcal{J})_d - \dim H_0(\mathcal{J})_d .$$



To compute $\boxed{\dim J(\gamma)_d}$ we consider the dual points in $\mathbb{P}^2$:

$$\ell_i = a_1\, x + a_2\, y + a_3\, z \quad \longleftrightarrow \quad P_i = [a_1 : a_2 : a_3] \in \mathbb{P}^2$$

Ex: $P_1 = [1 : 0 : 0], \quad P_2 = [0 : 1 : 0], \quad P_3 = [0 : 0 : 1]$.

Ideals of fat points

If $P = [a_0 : a_1 : \cdots : a_n] \in \mathbb{P}^n$, we can consider the ideal

$$\wp = \langle L_1, \ldots, L_n \rangle \quad \text{where } L_j(P) = 0$$

for independent linear forms L_j.

For a collection of points P_i and a collection of positive integers α_i
$I = \wp_1^{\alpha_1} \cap \cdots \cap \wp_t^{\alpha_t}$ is called an **ideal of fat points**. Via **apolarity**:

$$\boxed{\dim R/\langle \ell_1^{r+1}, \ldots, \ell_t^{r+1} \rangle_d = \dim R_d - \dim\left(I^{d-r}\right)_d}$$

- The expected dimension is $E(t, r+1, 3)_d = \left[\binom{d+2}{2} - t\binom{d-r+1}{2}\right]_+$

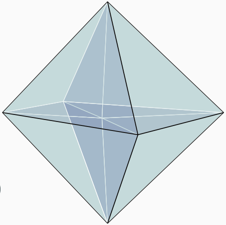
  In fact, $\dim(R/\langle \ell_1^{r+1}, \ldots, \ell_t^{r+1} \rangle)_d \geq F(t, r+1, 3)_d \geq E(t, r+1, 3)_d$.
- $F(t, r+1, n)_i$ was conjectured by Fröberg (1985) for **generic forms**.

– This formula is used to find bounds on the dimension of trivariate splines.

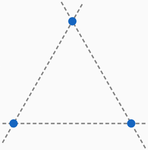
In $\mathbb{P}^2$ this conjecture became Segre-Harbourne-Gimigliano- Hirschowitz's conjecture (2001); Laface and Ugaglia in $\mathbb{P}^3$; Dumitrescu, Brambilla, and Postighel in $\mathbb{P}^n$.

Dual graphs

Ex: Regular octahedron

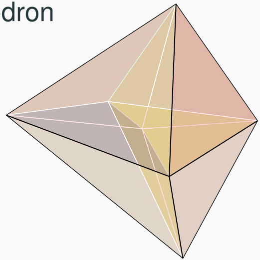


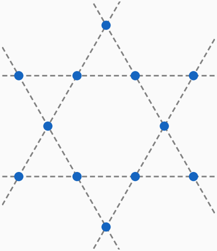
- Then 12 two-dimensional faces lie in the planes $\ell_1 = x$, $\ell_2 = y$ and $\ell_3 = z$.
- The dual points are $P_1 = (1,0,0)$, $P_2 = (0,1,0)$, $P_3 = (0,0,1)$



- Each interior edge τ lie in the intersection of 2 of these planes $\Rightarrow$ the corresponding 2 dual points lie on a line L_τ.

Ex: Generic octahedron



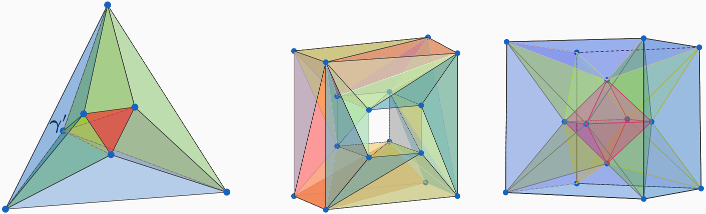


Combinatorial bounds on a fat point scheme

Thm (Cooper, Harbourne, and Teitler – 2011): Given a reduction vector $\boldsymbol{d} = (d_1, \ldots, d_n)$ they provide lower and upper bounds on the dimension of the fat points subscheme in $\mathbb{P}^2$.

Thm (Whiteley): For generic vertex stars $\mathcal{S}^r(\Delta)_d = \binom{d+2}{2}$ if $d \leq \frac{3r+1}{2}$.

Prop: For a vertex stars $\mathcal{J}(\gamma)_d = \mathcal{R}_d$ for $d > \frac{\widehat{\alpha}(I_X)r}{\widehat{\alpha}(I_X)-1}$, where $\widehat{\alpha}(I) = \lim_{s\to\infty} \frac{\alpha(I^{(s)})}{s}$ is the Waldschmidt constant.



A lower bound on the dimension of splines on tetrahedral vertex stars, SIAGA, 2021, & of tetrahedral splines in large degree, Constr Approx, 2024.

Some open questions

- **Conjecture** (splines on triangulations): $\dim \mathcal{S}_3^1(\Delta) = 3V_B + 2V_I + \sigma + 1$.
- Superspline spaces: $\mathcal{S}_d^{\boldsymbol{r}}(\Delta)$ –splines with mixed smoothness properties.

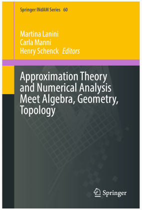
  For example, if $\tau = [\gamma, \gamma'] \in \Delta_1^\circ$, then $\mathcal{J}(\tau) = \left\langle \ell_\tau^{r_\tau + 1} \right\rangle \cap \mathfrak{m}_\gamma^{s_\gamma + 1} \cap \mathfrak{m}_{\gamma'}^{s_{\gamma'} + 1}$.

  When is $\mathcal{S}^{\boldsymbol{r}}(\Delta)$ a free module?

- Geometrically continuous splines: $G_d^1(\Delta, \Phi)$. Scheme structure?

  An algebraic framework for geometrically continuous splines, Math Comput, 2025.

- Survey/open questions:
  - The Algebra of Splines: Duality, Group Actions and Homology, Lanini, Schenck, and Tymoczko, 2024.
  - Some Problems at the Interface of Approximation Theory and Algebraic Geometry, Sottile, 2024.



Thank you!

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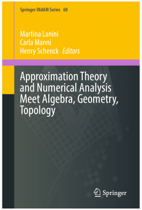

Thank you!