Final Projects

Directions: Work together in groups of 1-5 on the following four projects. We suggest, but do not insist, that you work in a group of > 1 people. For projects 1,2, and 3, your group should write a text file with (working!) code which answers the questions. Please provide documentation for your code so that we can understand what the code is doing. For project 4, your group should turn in a summary of observations and/or conjectures, but not necessarily code. Send your files via e-mail to peterson@math.colostate.edu *and* michael.dipasquale@colostate.edu. It is preferable that the work is turned in by Thursday night. However, at the very latest, please send your work by 12:00 p.m. (noon) on Friday, August 16.

1 Envelopes, Evolutes, and Euclidean Distance Degree

Read the section in Cox-Little-O'Shea's *Ideals, Varieties, and Algorithms* on *envelopes* of families of curves - this is in Chapter 3, section 4. We will expand slightly on the setup in Chapter 3 as follows. A *rational family of lines* is defined as a polynomial $L_t(x,y) \in \mathbb{Q}(t)[x,y]$ of the form

$$L_t(x,y) = \frac{1}{G(t)}(A(t)x + B(t)y + C(t)),$$

where $A(t), B(t), C(t)$, and $G(t)$ are polynomials in t. The envelope of $L_t(x,y)$ is the locus of x and y so that there is some t satisfying $L_t(x,y) = 0$ and $\frac{\partial L_t(x,y)}{\partial t} = 0$. As usual, we can handle this by introducing a new variable s which plays the role of $1/G(t)$. Now replace $1/G(t)$ by s in $L_t(x,y)$ and $\frac{\partial L_t(x,y)}{\partial t}$ to get the ideal generated by

$$\begin{gathered} s(A(t)x + B(t)y + C(t)) \\ s^2\left[(A'(t)G(t) - A(t)G'(t))x + (B'(t)G(t) - B(t)G'(t))y + (C'(t)G(t) - C(t)G'(t))\right] \\ sG(t) - 1 \end{gathered}$$

in the polynomial ring $\mathbb{Q}[s,t,x,y]$. You can obtain an equation for the envelope by eliminating s and t from this ideal.

Create a script in Macaulay2 which takes as input a rational family of lines and gives the equation of the envelope in $\mathbb{Q}[x,y]$ as output. One suggestion is to take as input the list $\{A(t), B(t), C(t), G(t)\}$ of polynomials in t where $L_t(x) = 1/G(t)(A(t)x + B(t)y + C(t))$. This will make it easier to define the list of equations above.

Now, consider a curve C in $\mathbb{R}^2$ parametrized by $(x(t), y(t))$. The rational family of lines

$$f_t(x,y) = (x'(t), y'(t)) \cdot (x - x(t), y - y(t)),$$

where $\cdot$ represents dot product, satisfies that $f_{t_0}(x,y) = 0$ is the equation of the *normal line* to C at the point $(x(t_0), y(t_0))$. The envelope of $f_t(x,y)$ is called the *evolute* of C.

Use your script for computing the envelope of a rational family of lines to compute the equation of the evolute of the following curves:

1. The parabola $y = x^2$ parametrized as $x(t) = t, y(t) = t^2$

2. The ellipse $x^2 + 4y^2 = 4$ parametrized as

$$x(t) = \frac{8t}{1+4t^2} \qquad y(t) = \frac{4t^2-1}{1+4t^2}$$

3. The cardioid $(x^2 + y^2 + x)^2 = x^2 + y^2$ parametrized as

$$x(t) = \frac{2t^2 - 2}{(1+t^2)^2} \qquad y(t) = \frac{-4t}{(1+t^2)^2}$$

Plot these evolutes in Sage along with their corresponding curves.

Finally, let C be the ellipse $x^2+4y^2 = 4$ parametrized above. Given a point $(p, q) \in \mathbb{R}^2$, consider the squared distance function $D(p, q) = (x-p)^2 + (y-q)^2$. Select several points (p, q) inside the evolute of the ellipse and find the number of *real* critical points of $D(p, q)$ restricted to C (use Lagrange multipliers and the Numerical Algebraic Geometry package for this). Now do the same for several points (p, q) *outside* of the evolute. Do you notice anything? Can you make some conjecture in the case of an ellipse?

2 Lines on Cubics

A classical result in Algebraic Geometry states that over an algebraically closed field, exactly 27 lines lie on a smooth projective cubic surface in $\mathbb{P}^3$. In this course, we have not discussed projective varieties or projective space. Thus, in this problem, we will consider how to find 27 lines lying on a "typical" smooth affine cubic surface. Let $F(x, y, z) \in \mathbb{Q}[x, y, z]$ be a cubic polynomial. Let V be the set of zeros of F in $\mathbb{C}^3$. Most lines in $\mathbb{C}^3$ have an ideal of the form $I = (x - (az + b), y - (cz + d))$ with $a, b, c, d \in \mathbb{C}$. We would like to find all conditions on a, b, c, d such that $F \in (x - (az + b), y - (cz + d))$. This amounts to making the substitution $x \to az + b, y \to cz + d$ and requiring that the result is zero. Making this substitution leads to a polynomial of the form $Az^3 + Bz^2 + Cz + D$ with $A, B, C, D \in \mathbb{Q}[a, b, c, d]$. The ideal $(A, B, C, D) \subseteq \mathbb{Q}[a, b, c, d]$ determines 27 points in $\mathbb{C}^4$. The coordinates of the points give values for a, b, c, d corresponding to the various lines.

- Write a Macaulay2 script that finds the 27 lines on a given cubic surface.

3 Homomorphisms of Polynomial Rings

The first part of this project is to code membership in the image of ring homomorphisms using elimination theory. Suppose $\phi : \mathbb{K}[w_1, \ldots, w_m] \to \mathbb{K}[x_1, \ldots, x_n]$ is a ring homomorphism defined by $\phi(w_i) = f_i$ for some polynomials $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_n]$.

Using Problem 7 on Problem Set 5 as a guide, write a Macaulay2 script which accepts as input a polynomial f and a list of polynomials $\{f_1, \ldots, f_m\}$, all taken from the ring $S = \mathbb{Q}[x_1, \ldots, x_n]$ which the user has previously defined. The $f_1, \ldots, f_m$ are the images of $w_1, \ldots, w_m$. From this input, your program should output a polynomial F in the w_i variables so that $\phi(F) = f$, if one exists, and otherwise should output the string "Not in the image of the ring map". Apply your code to answer the following:

- Write $x^{10} + y^{10} + z^{10}$ as a polynomial in the elementary symmetric functions $w_1 = x + y + z, w_2 = xy + xz + yz$ and $w_3 = xyz$.

- Decide whether the vector $v = (94, 144, 190)$ can be written as a linear combination $A(2, 3, 5) + B(3, 5, 7) + C(5, 7, 11) + D(7, 11, 13)$ with $A, B, C,$ and D all non-negative integers. If so, find such an $A, B, C,$ and D. Repeat with the vectors $v = (96, 146, 190)$ and $v = (95, 142, 190)$.

- Write the polynomial $f(x,y) = x^8 + 2x^6y^2 - x^5y^3 + 2x^4y^4 + x^3y^5 + 2x^2y^6 + y^8$ as a polynomial in $w_1 = x^2 + y^2, w_2 = x^3y - xy^3$, and $w_3 = x^2y^2$.

4 Singularities of Trigonometric curves

Suppose f is the implicit equation of a trigonometric rose with polar equation $r = \cos(\frac{n}{d}\theta)$ and $I = \langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$ is the ideal defining the singular locus of f. Make some observations and attempt a conjecture in answer to the following questions.

- Can you describe the singular points of $V(f)$ in terms of the parameters n and d?
- Can you describe the prime decomposition of $\sqrt{I}$ over $\mathbb{Q}$ in terms of the parameters n and d? What degrees does each prime have? In other words, how many points does each prime ideal define? (You can use the 'degree' command in Macaulay2 to compute this.)
- Can you describe the primary components of I over $\mathbb{Q}$ in terms of the parameters n and d?