Problem Set 3

Many of these problems are taken from the 4th edition of Cox-Little-O'Shea.

1. Consider the matrix $M = \begin{bmatrix} 1 & x & x^2 \\ x+1 & x^2+2 & x+y \end{bmatrix}$. Find the row reduced echelon form of M using Grobner basis commands in Macaulay2 (there is a script on the course website called 'Two ways to implement row reduced echelon form' indicating how this can be done). Do this for the fields below:

   $$F = \mathbb{Q}(x,y), F = \mathbf{frac}\left(\frac{\mathbb{Q}[x,y]}{\langle y^2 - x^3 - x - 1\rangle}\right), \text{ and } F = \mathbf{frac}\left(\frac{\mathbb{F}_7[x,y]}{\langle y^2 - x^3 - x - 1\rangle}\right).$$

   Recall if R is an integral domain then $\mathbf{frac}(R)$ denotes the field of fractions of R. The command "frac" in Macaulay2 will define fraction fields.

2. Use the division algorithm to divide $f = x^7y^2 + x^3y^2 - y + 1$ by the ordered list $L = (xy^2 - x, x - y^3)$, first using Graded Lex order (GLex in Macaulay2) and then using Lex order. Then switch the order of the two polynomials in L and repeat. For the computations in Lex order you may wish to use Macaulay2 - there is a script posted on the course website called 'Implementing polynomial division' which has a function for the division algorithm. Feel free to use it, but do at least one computation by hand to understand the algorithm.

3. In this problem we study the division of $f = x^3 - x^2y - x^2z + x$ by $f_1 = x^2y - z, f_2 = xy - 1$.

   (a) Using Graded Lex order, compute r_1 = remainder of f on division by (f_1, f_2) and r_2 = remainder of f on division by (f_2, f_1). (the results will be different!)

   (b) Is $r = r_1 - r_2$ in the ideal generated by f_1 and f_2?

   (c) Compute the remainder of r on division by (f_1, f_2).

   (d) Does the division algorithm give a solution to the ideal membership problem? In other words, if the remainder of a polynomial f on division by (f_1, f_2) is non-zero, does it mean that f is not in the ideal generated by f_1 and f_2?

   (e) Let $I = \langle f_1, f_2\rangle$. Show that the ideal $\langle \mathrm{LT}_{GLex}(f_1), \mathrm{LT}_{GLex}(f_2)\rangle$ is *not equal* to the ideal $\mathrm{LT}_{GLex}(I) = \langle \mathrm{LT}_{GLex}(f) \mid f \in I\rangle$.

   (f) Show that $I = \langle x - z, yz - 1\rangle$ and guess what $\mathrm{LT}_{GLex}(I)$ might be (after tomorrow you will be able to prove your guess).

4. Let $V \subset \mathbb{R}^3$ be parametrized by $x = t, y = t^3, z = t^4$. In the previous exercise set you showed that $I(V) = \langle y - x^3, z - x^4\rangle$. The crucial step is to show that any polynomial vanishing on V can be written as $h_1(y - x^3) + h_2(z - x^4)$. You will now prove this using the division algorithm.

   (a) Use the division algorithm to prove that *any* polynomial $f \in \mathbb{K}[x,y,z]$ can be written as

   $$f = h_1(y - x^3) + h_2(z - x^4) + r,$$

   where r is a polynomial involving x *only. Take care with what monomial order you choose!*

(b) Use (a) to prove that $I(V) = \langle y - x^3, z - x^4\rangle$, where V is the variety parametrized by $x = t, y = t^3, z = t^4$.

(c) Extend your results in (a) and (b) to find the ideal $I(V)$, where V is parametrized by $x = t, y = t^n, z = t^m$ (n, m are positive integers).

5. Consider the polynomial ring $\mathbb{C}[x, y]$ in two variables. Recall that we can regard this as an infinite dimensional vector space over $\mathbb{C}$ with basis $1, x, y, x^2, xy, y^2, \ldots$. Let $I = \langle x^2, y^2\rangle$.

   (a) Explain why $\mathbb{C}[x, y]/I$ is a vector space over $\mathbb{C}$.

   (b) Show that, as a vector space over $\mathbb{C}$, $\mathbb{C}[x, y]/I$ has a basis consisting of the monomials $B = \{1, x, y, xy\}$.

   (c) Let $F = axy + bx + cy + d$ be an element of $\mathbb{C}[x, y]/I$, where $a, b, c, d \in \mathbb{C}$. Then multiplication by F induces a map of $\mathbb{C}$-vector spaces $L_F : \mathbb{C}[x, y]/I \to \mathbb{C}[x, y]/I$. Find the matrix M_B for L_F with respect to the basis B.

   (d) Show that if $d = 0$ then the matrix M_B from part (b) is nilpotent (that is, show that $M_B^k = 0$ for some integer $k > 1$).

   (e) If you are familiar with Jordan canonical form, find the Jordan canonical form of M_B if $F = xy + x + y$.

6. Repeat parts (b) and (c) of problem 5 with $I = \langle x^2 + 2xy, y^2\rangle$ (the basis B will be the same).

7. In the text, an ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$ is called a monomial ideal if $I = \langle x^\alpha | \alpha \in A\rangle$ for some (possibly infinite) set $A \subset \mathbb{N}^n$. (Dickson's lemma shows that we can always take A to be a finite set.) Prove that $I \subset \mathbb{K}[x_1, \ldots, x_n]$ is a monomial ideal if and only if the following condition is satisfied: for any $f \in \mathbb{K}[x_1, \ldots, x_n]$, $f \in I$ if and only if every monomial of f is in I.

8. Suppose $I \subset \mathbb{K}[x_1, \ldots, x_n]$ is any ideal.

   (a) Explain why $\mathbb{K}[x_1, \ldots, x_n]/I$ is a $\mathbb{K}$-vector space.

   (b) If I is a monomial ideal, describe a (possibly infinite) basis for $\mathbb{K}[x_1, \ldots, x_n]/I$, and prove that your answer is correct. (Hint: look back at problem 5 part (b)).

9. Let $I = \langle x^2 + 2xy + 3y^2, x^2 + 6y^2, x^2 + xy + y^2\rangle$. We will consider the coefficients of I as coming from two different fields.

   (a) Show that $I \subset \mathbb{F}_{11}[x, y]$ is a monomial ideal.

   (b) Show that $I \subset \mathbb{F}_7[x, y]$ is not a monomial ideal.