Problem Set 4

Many of these problems are taken from the 4th edition of Cox-Little-O'Shea.

1. Consider the ideal $I = \langle x^2y - z, xy - 1\rangle$. Use Buchberger's algorithm to find a Gröbner basis for I with respect to Graded Lexicographic monomial order. Find a generating set for the ideal $\mathrm{LT}_{GLex}(I)$ of leading terms.

2. Use Buchberger's algorithm to compute a Gröbner basis for the following ideals with respect to Lex and GrLex orders. Compare your answers.

   (a) $I = \langle x^2 + y, x^4 + 2x^2y + y^2 + 3\rangle$

   (b) $I = \langle x - z^4, y - z^5\rangle$

3. GRevLex is not always better than Lex, but it often is. Use Macaulay2 to compute Gröbner bases for $I = \langle x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1\rangle$ with respect to GRevLex, with respect to Lex, and with respect to the monomial order defined by the weight vectors $[1, 0, 0], [1, 1, 1], [1, 1, 0]$. Compare your results.

4. (Multiplication matrices) Consider the ideal $I \subset \mathbb{C}[x, y]$ generated by

$$\begin{aligned} g_1 = &\quad x^2 - xy - 3x \\ g_2 = &\quad y^3 - y \\ g_3 = &\quad xy^2 + xy \end{aligned}$$

   We will find the affine variety $V(I) \subset \mathbb{C}^2$ and compare it to multiplication matrices on $\mathbb{C}[x, y]/I$.

   (a) The ideal $V(I)$ consists of a finite number of points. Find these points.

   (b) Use Buchberger's criterion to show that g_1, g_2, g_3 are a Gröbner basis with respect to Graded Reverse Lexicographic order (GRevLex). You can use the divisionAlgorithm function from the course website to divide the S-pairs.

   (c) Find the lead term ideal $\mathrm{LT}_{GRevLex}(I)$ and use this to find a monomial basis B for the quotient $\mathbb{C}[x, y]/I$.

   (d) As discussed in class, multiplication by a polynomial f induces a linear map $L_f : \mathbb{C}[x, y]/I \to \mathbb{C}[x, y]/I$. Find matrices M_x, M_y for the maps L_x and L_y using the basis B you found in part (c).

   (e) Find the eigenvalues of M_x and M_y. Do these seem to be related to the points of $V(I)$? See if you can formulate a conjecture. For more on this relationship, see Section 4 of Chapter 2 in the book *Using Algebraic Geometry*, by Cox, Little, and O'Shea.

5. (Minimal polynomials). Suppose you are given two algebraic numbers α, β with minimal polynomials $f(x), g(x) \in \mathbb{Z}[x]$. The following procedure allows you to find the minimal polynomial of $a\alpha + b\beta + c$.

   - Create a polynomial ring with $\mathbb{Q}[\alpha, \beta, s]$ (with Lex order and $\alpha > \beta > s$).
   - Create the ideal $I = \langle f(\alpha), g(\beta), s - (a\alpha + b\beta + c)\rangle$
   - Eliminate the variables α and β (in other words, find the elimination ideal I_2).

(a) Use this procedure to find the minimal polynomial of $\sqrt{3}+\sqrt{5}$.

(b) Notice that in the procedure outlined above that to produce the minimal polynomial of the sum of two algebraic numbers it is only necessary to specify the minimal polynomial, not the precise root which was taken. With this in mind, identify three other algebraic numbers that have the same minimal polynomial as $\sqrt{3}+\sqrt{5}$.

(c) Write a script in Macaulay2 which takes as input the minimal polynomials of two algebraic numbers and outputs the minimal polynomial of the sum of the two algebraic numbers. Can you modify your script to accept any linear combination of the two algebraic numbers?

For the following exercise, recall that the Bayer-Stillman ℓ elimination order on $\mathbb{K}[x_1,\ldots,x_n]$ is defined as follows: given monomials x^α, x^β, $x^\alpha > x^\beta$ if and only if $\alpha_1+\cdots+\alpha_\ell > \beta_1+\cdots+\beta_\ell$ or $\alpha_1+\cdots+\alpha_\ell = \beta_1+\cdots+\beta_\ell$ and $x^\alpha >_{GRevLex} x^\beta$. That is, compare monomials by first dotting their exponent vectors with the vector $(1,\ldots,1,0,\ldots,0)$ whose first ℓ coordinates are 1 and then break ties by using GRevLex order.

6. A monomial order $>$ on $\mathbb{K}[x_1,\ldots,x_n]$ is of ℓ *elimination type* provided that any monomial involving one of the variables $x_1,\ldots,x_\ell$ is greater than any monomial in $\mathbb{K}[x_{\ell+1},\ldots,x_n]$. If I is an ideal in $\mathbb{K}[x_1,\ldots,x_n]$ and G is a Gröbner basis of I with respect to a monomial order of ℓ-elimination type, prove that $G\cap\mathbb{K}[x_{\ell+1},\ldots,x_n]$ is a Gröbner basis for the ideal $I_\ell = I\cap\mathbb{K}[x_{\ell+1},\ldots,x_n]$. Deduce in particular that the Bayer-Stillman ℓ elimination order can be used to compute the elimination ideal I_ℓ.

7. (Symmetric Polynomials) Recall that the elementary symmetric polynomials in the variables a,b,c are $\sigma_1 = a+b+c, \sigma_2 = ab+ac+bc$, and $\sigma_3 = abc$.

   (a) Use elimination in the polynomial ring $\mathbb{Q}[a,b,c,\sigma_1,\sigma_2,\sigma_3]$ to write $a^4+b^4+c^4$ as a polynomial in the elementary symmetric polynomials $\sigma_1,\sigma_2,\sigma_3$. Hint: eliminate a, b, and c from an appropriately defined ideal.

   (b) Using the same strategy as in (a), write $(a-b)^2(a-c)^2(b-c)^2$ as a polynomial in $\sigma_1,\sigma_2,\sigma_3$.

   (c) Recall that if a cubic polynomial, F, has roots a,b,c then the polynomial can be written as $F = x^3-\sigma_1x^2+\sigma_2x-\sigma_3$. In the polynomial ring $\mathbb{Q}[x,\sigma_1,\sigma_2,\sigma_3]$ with Lex order, try eliminating x from the ideal $\langle F, \frac{\partial F}{\partial x}\rangle$ (this is the first elimination ideal). Do you notice any similarities with the answer from part (b)?

8. (4-bar linkage) Find the implicit equation of the coupler curve for a 4-bar linkage with base at $(0,0),(5,0)$ and arm lengths $(1,2,4,3,4)$. See Figure 1 for the configuration of lengths. To get this equation, set up a polynomial ring in the six variables corresponding to the x and y coordinates of the three movable joints, as in Figure 1. Then set up an ideal whose generators correspond to the lengths of the bars. Then eliminate all variables except s and t, the variables corresponding to the tip of the triangle. Check out the following website for some nice visualizations of coupler curves of 4-bar mechanisms: https://saltire.com/HTML5/Mechanisms/linkages.html.

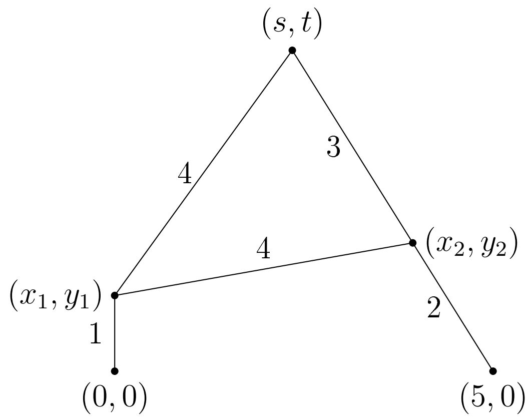

Figure 1: 4-bar linkage for problem 8

9. (Ideal intersection, ideal quotient)

   (a) Let $I = \langle y^2 - xz, xy - z, x^2 - y \rangle$ and let $J = \langle x - 1, y - 2, z - 3 \rangle$. Use elimination to compute $(tI + (1 - t)J) \cap \mathbb{Q}[x, y, z]$. Compare your result to the output of the command intersect(I, J) in Macaulay2.

   (b) Let L be any degree 1 polynomial from the ideal J. Compute $(I \cap J) : L$ and compare your result to the ideal I.