Problem Set 6

Many of these problems are taken from the 4th edition of Cox-Little-O'Shea.

1. (Implicitization) Use the method discussed in class to get a polynomial equation for the trigonometric rose with polar equation $r = \cos(\frac{n}{d}\theta)$ for $(n, d) = (3, 2), (1, 4), (2, 7)$, and $(7, 4)$. Use Sage to plot your result.

2. For each of the polynomials f that you obtained in Problem 1, define the ideal $I = \langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$. The *number* of singular points of $V(f)$ can be computed as the vector space dimension of $\mathbb{Q}[x, y]/\sqrt{I}$. In Macaulay2 this can be computed using the commands 'degree radical I'. See if you can get the same number by counting the singular points in the plots you obtained in Problem 1.

3. Find the implicit equation f of the rose defined by $r = \cos(\frac{1}{2}d)$. Now choose a point $(a, b) \in \mathbb{R}^2$ (pick integer coordinates) and let D be the squared distance function from (a, b), i.e. $D = (x - a)^2 + (y - b)^2$. Set up the system of equations coming from Lagrange multipliers whose solutions are the critical points of D on $V(f)$. Count the number of critical points of D on $V(f)$ by using the 'degree' command in Macaulay2 as follows:

   (a) Define the ideal J associated to the system of equations you got from Lagrange multipliers.

   (b) Eliminate the variable associated to the multiplier λ from J. Call the resulting ideal I.

   (c) Use the command 'degree radical I' to compute the number of critical points c of D restricted to $V(f)$.

   (d) Do you expect the singular points of f to be critical points? Let s be the number of singular points of $V(f)$ *which are also critical points* and compute the number $c - s$.

   (e) Try computing $c - s$ for different points (a, b). Do you get the same thing? This number is related to the *Euclidean distance degree.*
   See https://arxiv.org/pdf/1309.0049.pdf for more.

4. (Primary ideal basics)

   (a) Show that the radical of a primary ideal is a prime ideal. If I is primary and $\sqrt{I} = \mathfrak{p}$, we say I is $\mathfrak{p}$-primary.

   (b) Show that if I and J are $\mathfrak{p}$-primary then $I \cap J$ is $\mathfrak{p}$-primary.

5. (Radical ideal basics)

   (a) Show that if I is a radical ideal and if J is any ideal then $I : J$ is a radical ideal.

   (b) Show that $\sqrt{I : J^\infty} = \sqrt{I} : J$

   (c) Show that $\sqrt{IJ} = \sqrt{I \cap J}$

   (d) A monomial is *squarefree* if it is not divisible by the square of any variable. Show that a monomial ideal is radical $\iff$ the minimal generators of I are squarefree.

(e) Show that if $I \subseteq \sqrt{J}$ then there exists an m such that $I^m \subseteq J$

6. (Exploring primary decomposition, computationally) Carry out the steps below for the two ideals:

   - $I = \langle x^3, xy^2, xz^2 \rangle$
   - $I = \langle xyz, x^2z, xy^2, x^2y, x^3 \rangle$.

   (a) Use the command 'primaryDecomposition' to get a primary decomposition of I. Check the answer that Macaulay2 gives you: verify that each of the ideals is primary and that they intersect to give I. The Macaulay2 command 'intersect' can be useful here.

   (b) Use the command 'associatedPrimes' to get the associated primes of I.

   (c) The *minimal primes* of I are the associated primes of I that are minimal with respect to inclusion. Use the command 'minimalPrimes' to get the minimal primes of I. Verify that these are the associated primes of the radical I.

   (d) It is a theorem that the associated primes of an ideal I are the maximal proper ideals of the form $I : f$, where f runs across all polynomials. For each of the associated primes P in part (b), see if you can find an element f in the polynomial ring $\mathbb{Q}[x, y, z]$ so that $I : f = P$. In Macaulay2, if I is an ideal and f is a polynomial, then $I : f$ is computed literally by typing 'I:f'.

7. (Weak Nullstellensatz and maximal ideals) Consider the following two statements in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ over an algebraically closed field $\mathbb{K}$.

   (a) (Weak Nullstellensatz) $V(I) \neq \emptyset \Leftrightarrow I = \mathbb{K}[x_1, \ldots, x_n]$.

   (b) Every maximal ideal $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ has the form $I = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$ for some $a_1, \ldots, a_n \in \mathbb{K}$.

   In class we showed 7a⇒7b. Prove that 7b⇒7a. In other words, prove that the description of maximal ideals in 7b is equivalent to the weak Nullstellensatz.

8. (Exploring maximal ideals) Suppose $f_1(x), \ldots, f_n(x) \in \mathbb{K}[x]$. In the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ (where $\mathbb{K}$ is any field), consider the ideal

$$I = \langle f_1(x_1), x_2 - f_2(x_1), \ldots, x_n - f_n(x_1) \rangle$$

   (a) Show that every $f \in \mathbb{K}[x_1, \ldots, x_n]$ can be written uniquely as $f = q + r$ where $q \in I$ and $r \in \mathbb{K}[x_1]$ with either $r = 0$ or $\deg(r) < \deg(f_1)$. Hint: use Lex order with x_1 the smallest variable (instead of the largest).

   (b) Use part (a) to prove that $\mathbb{K}[x_1, \ldots, x_n]/I \cong \mathbb{K}[x]/\langle f_1(x) \rangle$.

   (c) Prove that the following are equivalent:

      i. I is prime.

      ii. I is maximal.

      iii. $f_1(x)$ is irreducible.

   (d) Prove that I is radical if and only if $f_1(x)$ is squarefree.

Remark 1. *It can be shown that* **every** *maximal ideal in the polynomial ring* $\mathbb{K}[x_1, \ldots, x_n]$ *has the form of the ideal* I *in Problem 8 for some polynomial* $f_1(x)$ *which is irreducible over* $\mathbb{K}$.

9. (Squarefree lead term ideals) There is a general philosophy that good properties of an ideal I cannot be *gained* when passing to the lead term ideal $\langle \mathrm{LT}(I) \rangle$, only *lost*. In this problem you will prove one instance of this philosophy. Suppose $I \subset \mathbb{K}[x_1, \ldots, x_n]$ is an ideal and $G = \{g_1, \ldots, g_r\}$ is a Gröbner basis for I satisfying that $\mathrm{LT}(g_i)$ is squarefree for $i = 1, \ldots, r$.

   (a) If $f \in \sqrt{I}$, prove that $\mathrm{LT}(f)$ is divisible by $\mathrm{LT}(g_i)$ for some $g_i \in G$. Hint: $f^r \in I$ for some r.

   (b) Prove that I is radical. Hint: show that G is a Gröbner basis for $\sqrt{I}$.

   (c) From (a) and (b), conclude that if $\langle \mathrm{LT}(I) \rangle$ is radical, then I is radical.

   (d) Find an example to show that if I is radical, it is not necessarily true that $\langle \mathrm{LT}(I) \rangle$ is radical.

10. (Prime ideal basics)

   (a) Show that an ideal P is prime if and only if for any two ideals I, J, $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$.

   (b) Show that if $I_1, \ldots, I_k$ are ideals and P is prime, then $\cap_{i=1}^k I_i \subseteq P$ if and only if $I_j \subseteq P$ for some j.

   (c) (Prime avoidance) If $P_1, \ldots, P_k$ are prime ideals and $I \subseteq \cup_{i=1}^k P_i$ then $I \subseteq P_j$ for some j. Hint: use induction on k.

11. (Optional: Integer linear programming using Gröbner bases) *If you are interested in the ideas behind this problem you should consult Chapter 8, Section 1 of Using Algebraic Geometry by Cox, Little, and O'Shea.* **This problem builds on problems 7 and 8 from Problem Set 5.**

   The central problem of *integer linear programming* is to minimize a linear function $\ell(A_1, \ldots, A_n) = c_1 A_1 + \cdots + c_n A_n$ subject to a system of *constraint* equations:

$$\begin{array}{rl} a_{11}A_1 + a_{12}A_2 + \cdots + a_{1n}A_n = & b_1 \\ a_{21}A_1 + a_{22}A_2 + \cdots + a_{2n}A_n = & b_2 \\ \vdots & \vdots \\ a_{m1}A_1 + a_{n2}A_2 + \cdots + a_{mn}A_n = & b_m \end{array},$$

   where $A_1, \ldots, A_n \in \mathbb{Z}_{\geq 0}$. The *feasible region* is the set of tuples $(A_1, \ldots, A_n) \in \mathbb{R}^n_{\geq 0}$ satisfying the constraint equations.

   For simplicity, we will restrict to the case $a_i j \geq 0$ **and** $b_i \geq 0$ **for** $1 \leq i \leq m, 1 \leq j \leq n$. Define the map of polynomial rings $\phi : \mathbb{K}[w_1, \ldots, w_n] \to \mathbb{K}[z_1, \ldots, z_m]$ by $\phi(w_j) = \prod_{i=1}^m z_i^{a_{ij}}$.

   Let $S = \mathbb{K}[z_1, \ldots, z_m, w_1, \ldots, w_n]$ and define the monomial order $<_\ell$ on S as follows: first, compare monomials using the vector $[1, \ldots, 1, 0, \ldots, 0]$ which has 1 in the first m entries and 0 in the last n entries, then break ties using the vector

$[0, \ldots, 0, c_1, \ldots, c_m]$ which has 0 in the first n entries, then finally break remaining ties using GRevLex order. (This is a slight adjustment of a Bayer-Stillman ℓ-elimination order). The order $<_\ell$ is called an *adapted* order for the linear programming problem.

(a) Prove that $(A_1, \ldots, A_n) \in \mathbb{Z}_{\geq 0}$ is in the feasible region if and only if

$$\phi(w_1^{A_1} w_2^{A_2} \cdots w_n^{A_n}) = z_1^{b_1} z_2^{b_2} \cdots z_m^{b_m}$$

Conclude that the integer points of the feasible region are in 1-1 correspondence with monomials that map to $z_1^{b_1} \cdots z_m^{b_m}$ under ϕ.

(b) Define the ideal $I = \langle w_i - \phi(w_i) \mid i = 1, \ldots, n \rangle$ and let G be a Gröbner basis for I with the adapted monomial order $<_\ell$ from above. Prove that the remainder R of $z_1^{b_1} \ldots z_m^{b_m}$ under division by the Gröbner basis G gives a solution to the linear programming problem above (namely, minimize ℓ over non-negative integers subject to the constraint equations above).

(c) Consider the integer programming problem: minimize

$$\ell(A_1, A_2) = A_1 + 2A_2 + 3A_3$$

subject to

$$\begin{aligned} 2A_1 + 3A_2 + 3A_3 &= 300 \\ 5A_1 + A_2 + 3A_3 &= 300, \end{aligned}$$

with $A_1, A_2, A_3 \in \mathbb{Z}_{\geq 0}$. Solve this problem in Macaulay2 using Gröbner bases as indicated in (b). Do you get different answers if you change the objective function $\ell(A_1, A_2)$? You may want to refer to Problem 8 on Problem Set 5 for useful Macaulay2 commands.

(d) The linear programming problem in (c) is small enough to solve by hand using a bit of linear algebra. See if you can get the same answer that you obtained in (b).

(e) Now use Macaulay2 to minimize

$$\ell(A, B, C, D) = 2A + 3B + C + 5D$$

subject to

$$\begin{aligned} 3A + 2B + C + D &= 10 \\ 4A + B + C &= 5, \end{aligned}$$

with $A, B, C, D \in \mathbb{Z}_{\geq 0}$. Then change the right hand sides of the equations to 20 and 14 and redo the computation. (This problem comes from Section 8.1 of Using Algebraic Geometry).

12. (Open-ended problem) Suppose f is the equation of a trigonometric rose from Problem 1. Can you describe the singular points of $V(f)$ in terms of the parameters n, d? What about the associated primes of $\sqrt{I}$ where $I = \langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ over $\mathbb{Q}$? What about the associated primes of I over $\mathbb{Q}$? Explore using Macaulay2!

13. (Open-ended problem) If f is a polynomial in three variables, describe a way to compute the critical points of f restricted to the sphere $x^2 + y^2 + z^2 = 1$.