Problem Set 7

One of these problems is taken from the 4th edition of Cox-Little-O'Shea.

1. Let $I = \langle x^3 - x, y^3 - 5y^2 + 6y, x(y - 3x - 3)(2x + y - 1)\rangle \subset R = \mathbb{Q}[x, y]$. With the monomial order GLex with $x > y > z$, use Macaulay2 to find bases of $(R/I)_{\leq 0}, (R/I)_{\leq 1}, (R/I)_{\leq 2}$, and $(R/I)_{\leq 3}$. By appealing to a 'staircase diagram' for monomials, find the affine Hilbert function of R/I. What is the affine Hilbert polynomial of R/I? What is the dimension of R/I?

2. Find the affine Hilbert function $HF^a_{R/I}(t)$ for $t = 0, 1, 2, 3$ for the choices of R and I listed below. Use a graded order of your choice (the default GRevLex order is great).

   (a) $R = \mathbb{Q}[x, y], I = \langle x^3y, xy^2\rangle$

   (b) $R = \mathbb{Q}[x, y, z], I = \langle x^3 - yz^2, y^4 - x^2yz\rangle$

   (c) $R = \mathbb{Q}[x, y, z, w], I = \langle x^2 - yw, y^3 - xz, yw - xw, yz^2 + xw\rangle$

   Recall that $\dim(R/I)$ is the degree of the affine Hilbert Polynomial $HP^a_{R/I}(d)$. In each of (a),(b),(c) above, print enough values of the affine Hilbert function to observe the rate of growth of the Hilbert polynomial. Use this to determine $\dim(R/I)$ for each of these ideals. (Recall that taking successive differences in a sequence with polynomial growth reveals the degree of the polynomial.)

Once you are finished, feel free to start working on the assigned Macaulay2 projects.