SUPPLEMENTARY COMPUTATIONS FOR *INEQUALITIES FOR FREE MULTI-BRAID ARRANGEMENTS* SUBMITTED TO PROCEEDINGS OF THE JAPAN ACADEMY SERIES A

In this note we provide details for a straightforward but tedious computation involved in the proof of Proposition 5.5 in *Inequalities for Free Multi-Braid Arrangements*, submitted to the Proceedings of the Japan Academy, Series A. For completeness, we recall the notation from that paper.

- $S = \mathbb{K}[x_0, \dots, x_\ell]$, where $\mathbb{K}$ has characteristic zero
- H_{ij} is the hyperplane defined by $x_i - x_j = 0$
- $A_\ell = \cup_{0 \le i < j \le \ell} H_{ij}$ denotes the braid arrangement of rank ℓ
- $K_{\ell+1}$ denotes the complete graph on vertices $\{v_0, \dots, v_\ell\}$; the edge $\{v_i, v_j\}$ in $K_{\ell+1}$ corresponds to the hyperplane $V(x_i - x_j)$ in A_ℓ.
- $\mathbf{m} : A_\ell \to \mathbb{Z}_{>0}$ denotes a map associating to each hyperplane in $\mathcal{A}_\ell$ (equivalently each edge in $K_{\ell+1}$) a positive integer. We write m_{ij} for $\mathbf{m}(H_{ij})$.
- We denote by Λ^b_ℓ the *balanced cone* of multiplicities on A_ℓ; this is the set of multiplicities $\mathbf{m} \in \mathbb{N}^{\binom{\ell+1}{2}}$ satisfying $m_{ij} \le m_{ik} + m_{jk} + 1$ for every triple of distinct integers i, j, k between 0 and ℓ.
- Given $\mathbf{m} \in \Lambda^b_\ell$, we call $\mathbf{m}$ an ANN multiplicity if there are non-negative integers $n_0, \dots, n_\ell$ and $\epsilon_{ij} \in \{-1, 0, 1\}$ for every $0 \le i < j \le \ell$ so that $m_{ij} = n_i + n_j + \epsilon_{ij}$ for every $0 \le i < j \le \ell$.
- $C = \{\{v_i, v_j\}, \{v_j, v_s\}, \{v_s, v_t\}, \{v_i, v_t\}\}$ represents a four-cycle in $K_{\ell+1}$ and $\mathbf{m}(C)$ denotes the quantity $|m_{ij} - m_{js} + m_{st} - m_{it}|$
- Given $U \subset \{v_0, \dots, v_\ell\}$ with $|U| \ge 4$:
  - $\mathbf{m}_U$ is the restriction of $\mathbf{m}$ to those hyperplanes H_{ij} with $\{v_i, v_j\} \subset U$.
  - For a four-cycle C, we write $C \subset U$ if all vertices of C are in U.
  - $\mathrm{DV}(\mathbf{m}_U) := \sum_{C \subset U} \mathbf{m}(C)^2$, where the sum is over all four-cycles $C \subset U$.
  - q_U denotes the number of *odd three-cycles* of $\mathbf{m}$ contained in U; that is q_U is the number of subsets $\{i, j, k\} \subset U$ so that $m_{ij} + m_{ik} + m_{jk}$ is odd.

We now present the notion of signed-eliminable graphs from [3, 1]. Let G be a *signed* graph on $\ell + 1$ vertices. That is, each edge of G is assigned either a $+$ or a $-$, and so the edge set E_G decomposes as a disjoint union $E_G = E^+_G \cup E^-_G$. Define

$$m_G(ij) = \begin{cases} 1 & \{i,j\} \in E^+_G \\ -1 & \{i,j\} \in E^-_G \\ 0 & \text{otherwise.} \end{cases}$$

The graph G is *signed-eliminable* with *signed-elimination ordering* $\nu : V(G) \to \{0, \dots, \ell\}$ if ν is bijective and, for every three vertices $v_i, v_j, v_k \in V(G)$ with $\nu(v_i), \nu(v_j) < \nu(v_k)$, the induced sub-graph $G|_{v_i, v_j, v_k}$ satisfies the following conditions.

- For $\sigma \in \{+, -\}$, if $\{v_i, v_k\}$ and $\{v_j, v_k\}$ are edges in E^σ_G then $\{v_i, v_j\} \in E^\sigma_G$
- For $\sigma \in \{+, -\}$, if $\{v_k, v_i\} \in E^\sigma_G$ and $\{v_i, v_j\} \in E^{-\sigma}_G$ then $\{v_k, v_j\} \in E_G$

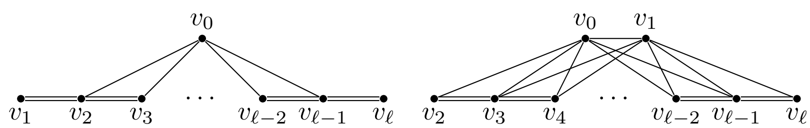


FIGURE 1. σ-mountain (at left) and σ-hill (at right)

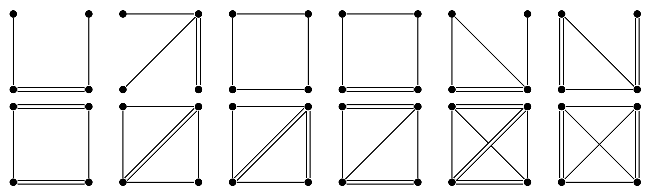


TABLE 1. Graphs on four vertices which are *not* signed-eliminable

These two conditions generalize the notion of a graph possessing an elimination ordering, which is equivalent to the graph being chordal. A graph is chordal if and only if it has no induced sub-graph which is a cycle of length at least four. In [3], Nuida establishes a similar characterization for signed-eliminable graphs, to which we now turn.

Definition 0.1. (1) A graph with $(\ell+1)$ vertices $v_0, v_1, \ldots, v_\ell$ with $\ell \geq 3$ is a σ-mountain, where $\sigma \in \{+,-\}$, if $\{v_0, v_i\} \in E_G^\sigma$ for $i = 2, \ldots, \ell-1$, $\{v_i, v_{i+1}\} \in E_G^{-\sigma}$ for $i = 1, \ldots, \ell-1$, and no other pair of vertices is joined by an edge. (See Figure 1 - edges of sign σ are denoted by a single edge and edges of sign $-\sigma$ are denoted by a doubled edge.)

(2) A graph with $(\ell+1)$ vertices $v_0, v_1, v_2, \ldots, v_\ell$ with $\ell \geq 3$ is a σ-hill, where $\sigma \in \{+,-\}$, if $\{v_0, v_1\} \in E_G^\sigma$, $\{v_0, v_i\} \in E_G^\sigma$ for $i = 2, \ldots, \ell-1$, $\{v_1, v_i\} \in E_G^\sigma$ for $i = 3, \ldots, \ell$, $\{v_i, v_{i+1}\} \in E_G^{-\sigma}$ for $i = 2, \ldots, \ell-1$, and no other pair of vertices is connected by an edge. (See Figure 1.)

(3) A graph with $(\ell+1)$ vertices $v_0, \ldots, v_\ell$ with $\ell \geq 2$ is a σ-cycle if $\{v_i, v_{i+1}\} \in E_G^\sigma$ for $i = 0, \ldots, \ell-1$, $\{v_0, v_\ell\} \in E_G^\sigma$, and no other pair of vertices is connected by an edge.

Theorem 0.2. [3, Theorem 5.1] *Let G be a signed graph. Then G is signed-eliminable if and only if the following three conditions are satisfied.*

($C1$) *Both G_+ and G_- are chordal.*
($C2$) *Every induced sub-graph on four vertices is signed eliminable.*
($C3$) *No induced sub-graph of G is a σ-mountain or a σ-hill.*

All signed-eliminable graphs on four vertices are listed (with an elimination ordering) in [1, Example 2.1], along with those which are not signed-eliminable. For use in the proof of Corollary 0.3, we also list those graphs which are not signed-eliminable in Table 1. The property of being signed-eliminable is preserved under interchanging $+$ and $-$. Consequently, we list these graphs in Table 1 up to automorphism with the convention that a single edge takes one of the signs $+, -$, while a double edge takes the other sign.

Corollary 0.3. *Let G be a signed graph. Then G is signed-eliminable if and only if the following three conditions are satisfied.*

($C1'$) *No induced sub-graph of G is a σ-cycle of length > 3.*
($C2$) *Every induced sub-graph on four vertices is signed eliminable.*
($C3$) *No induced sub-graph of G is a σ-mountain or a σ-hill.*

Proof. Clearly ($C1$) from Theorem 0.2 implies ($C1'$). We show that ($C1'$) and ($C2$) imply condition ($C1$). Assume for contradiction that E^σ_G, $\sigma \in \{-,+\}$, is a cycle of length $\ell+1 > 3$ and $V_G = \{v_0, \ldots, v_\ell\}$ where $\{v_i, v_{i+1}\} \in E^\sigma_G$ for $i = 0, \ldots, \ell-1$ and $\{v_0, v_\ell\} \in E^\sigma_G$. If $E^{-\sigma}_G = \emptyset$ then G is a σ-cycle which is forbidden by ($C1'$), so we assume $E^{-\sigma}_G \neq \emptyset$. Let m be the maximal integer so that there is a sequence of consecutive vertices $v_i, v_{i+1} \ldots, v_{i+m-1}$ so that the induced sub-graph on these consecutive vertices consists only of edges in E^σ_G. Since $E^{-\sigma}_G \neq \emptyset$, $m < \ell+1$. Relabel the vertices so that $v_0, \ldots, v_{m-1}$ are the vertices of a maximal induced sub-graph with edges only in E^σ_G. If $m = 2$ or $m = 3$, then the induced sub-graph on v_0, v_1, v_2, v_3 consists of the three σ edges $\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}$ along with at least one $-\sigma$ edge. No such graph is signed eliminable (see Table 1). So $m \geq 4$. Now consider the induced sub-graph H on $v_0, v_1, \ldots, v_{m-1}, v_m$. By definition of m, H has exactly one $-\sigma$ edge, namely $\{v_0, v_m\}$. But then the induced sub-graph on v_0, v_1, v_{m-1}, v_m consists of the two σ edges $\{v_0, v_1\}, \{v_{m-1}, v_m\}$ and the $-\sigma$ edge $\{v_0, v_m\}$, which is not signed-eliminable. It follows that E^σ_G cannot have a cycle of length > 3, so E^σ_G is chordal. □

The following is Proposition 5.5 in 'Inequalities for Free Multi-Braid Arrangements.' We give a number of tables which makes the somewhat tedious computations clearer.

Proposition 0.4. *Suppose $n_0, \ldots, n_\ell$ are non-negative integers, G is a signed graph on $v_0, \ldots, v_\ell$, and let $\mathbf{m}$ be the multiplicity on A_ℓ given by $m_{ij} = n_i + n_j + m_G(ij)$. If G is not signed-eliminable, then there is a subset $U \subset \{0, \ldots, \ell\}$ so that $DV(\mathbf{m}_U) > q_U \cdot (|U| - 1)$.*

Proof. Notice that, for a four-cycle traversing i, j, s, t in order,

$$\mathbf{m}(C) = |m_{ij} - m_{js} + m_{st} - m_{it}| = |m_G(ij) - m_G(js) + m_G(st) - m_G(it)|.$$

Furthermore, for a three-cycle $\{i, j, k\}$,

$$m_{ij} + m_{ik} + m_{jk} = 2(n_i + n_j + n_k) + m_G(ij) + m_G(ik) + m_G(jk).$$

It follows that the values of $\mathrm{DV}(\mathbf{m}_U) = \sum_{C \subset U} \mathbf{m}(C)^2$ and $q_U = \#$ odd three cycles of $\mathbf{m}$ in U may be determined after replacing m_{ij} by $m_G(ij)$, which takes values only in $\{-1, 0, 1\}$. Hereafter we write $\mathrm{DV}(G)$ for $\mathrm{DV}(\mathbf{m})$ and q_G for q to emphasize their dependence only on the signed graph G. If $U \subset \{v_0, \ldots, v_\ell\}$, we let $\mathrm{DV}(G_U)$ represent $\mathrm{DV}(\mathbf{m}_U)$ to emphasize dependence only on G and the subset U.

Now, if G is not signed eliminable then by Corollary 0.3 G contains an induced sub-graph H which is

- a signed graph on four vertices which is not signed-eliminable,
- a σ-cycle of length > 3,
- a σ-hill,
- or a σ-mountain.

We assume $G = H$ and show that $DV(G) > q_G \ell$ in each of these cases, where ℓ is one less than the number of vertices of G. The inequality $DV(G) > 3q_G$ can easily be verified by hand for each of the twelve graphs on four vertices which are not signed-eliminable (see Table 1); this is also done in [2, Corollary 6.2]. If G is a σ-cycle, σ-mountain, or σ-hill on $(\ell+1)$ vertices we will show that $\mathrm{DV}(G)$ and q_G are given by the formulas:

$$\mathrm{DV}(G) = \ell^3 - 2\ell^2 - \ell + 2 \tag{1}$$

$$q_G = \ell^2 - 2\ell - 3. \tag{2}$$

Given these formulas, note that $\mathrm{DV}(G) = q\ell + 2(\ell+1) > q\ell$, thus proving the result. We prove Equations (1) and (2) for the σ-cycle directly, relying on the two additional formulas:

$$\mathrm{DV}(G) = \sum_{U \subset V_G, |U|=4} \mathrm{DV}(G_U) \tag{3}$$

$$q_G = \Big(\sum_{U \subset V_G, |U|=4} q_U\Big)/(\ell - 2). \tag{4}$$

Equation (3) follows since each four-cycle is contained in a unique induced sub-graph on four vertices and Equation (4) follows since each three-cycle appears in $(\ell-2)$ sub-graphs on four vertices. Using these equations, it suffices to identify all possible types of induced sub-graphs of the σ-cycle on four vertices, how many of each type there are, and compute $\mathrm{DV}(G_U)$ and q_U for each of these. Then we use Equation (3) to compute $\mathrm{DV}(G)$ and Equation (4) to compute q_G.

The list of all possible induced sub-graphs with four vertices of a σ-cycle on $(\ell+1)$ vertices are listed in Table 2. The number of sub-graphs of each type is listed in the second column, while the third and fourth columns record q_U and $\mathrm{DV}(G_U)$, respectively, for each type of sub-graph. The final row records the total number of sub-graphs on four vertices, the number of odd three-cycles, and the deviation of $\mathbf{m}$, $\mathrm{DV}(\mathbf{m}) = \sum_C \mathbf{m}(C)^2$. We find that $\mathrm{DV}(\mathbf{m}) = \ell^3 - 2\ell^2 - \ell + 2$ and $q = \ell^2 - 2\ell - 3$, proving Equations (1) and (2) for the σ-cycle.

The same computations can be done to prove Equations (1) and (2) for the σ-mountain and σ-hill; the corresponding tables are Table 3 and Table 4, respectively. Due to their length we give each table its own page (or two pages). □

σ-cycle of length $(\ell+1)$

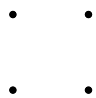
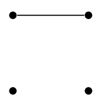
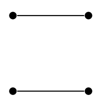
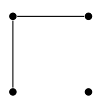
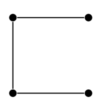
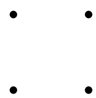
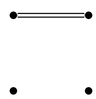
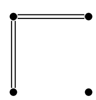
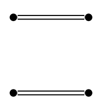
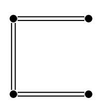
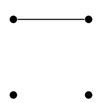
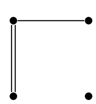
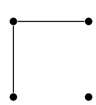
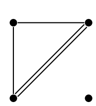
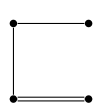
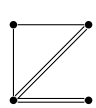
Type of sub-graph	Count	q_U	$\mathrm{DV}(G_U)$
	$\binom{\ell-4}{2}+\binom{\ell-3}{2}$	0	0
	$(\ell+1)\binom{\ell-4}{2}$	2	2
	$\frac{(\ell+1)(\ell-4)}{2}$	4	8
	$(\ell+1)(\ell-4)$	2	2
	$\ell+1$	2	6
Total	$\binom{\ell+1}{4}$	$q=\ell^2-2\ell-3$	$\mathrm{DV}=\ell^3-2\ell^2-\ell+2$

Table 2: Computing $\mathrm{DV}(G)$ where G is a σ-cycle

σ-mountain on $(\ell+1)$ vertices

Type of subgraph	Count	q_U	$\mathrm{DV}(G_U)$
	$\binom{\ell-3}{4}$	0	0
	$3\binom{\ell-3}{3}$	2	2
	$2\binom{\ell-3}{2}$	2	2
	$2\ell-9+\binom{\ell-5}{2}$	4	8
	$(\ell-3)$	2	6
	$\ell-4$	2	2
	2	2	6
	$2\binom{\ell-4}{2}$	2	2
	$2(\ell-4)$	4	8
	$2(\ell-4)$	2	2
	2	2	6

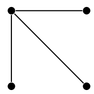
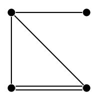
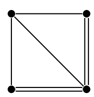
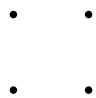
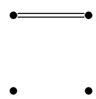
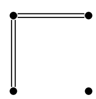
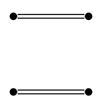
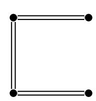
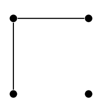
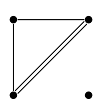
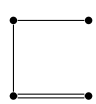
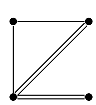
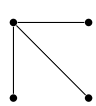
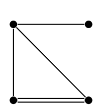
	$\binom{\ell-4}{3}$	0	0
	$2\binom{\ell-4}{2}$	2	2
	$\ell-4$	2	2
Total	$\binom{\ell+1}{4}$	$q_G = \ell^2 - 2\ell - 3$	$\mathrm{DV} = \ell^3 - 2\ell^2 - \ell + 2$

Table 3: Computing $\mathrm{DV}(G)$ where G is a σ-mountain

σ-hill on $(\ell+1)$ vertices

Type of subgraph	Count	q_U	$\mathrm{DV}(G_U)$
	$\binom{\ell-4}{4}$	0	0
	$3\binom{\ell-4}{3}$	2	2
	$2\binom{\ell-4}{2}$	2	2
	$2\ell-11+\binom{\ell-6}{2}$	4	8
	$\ell-4$	2	6
	$2\binom{\ell-4}{2}$	2	2
	$2(\ell-4)$	4	8
	$2(\ell-4)$	2	2
	2	2	6
	$2\binom{\ell-4}{3}$	0	0
	$4\binom{\ell-4}{2}$	2	2

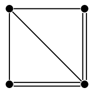
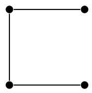
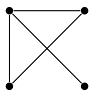
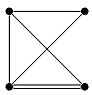
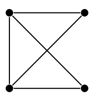
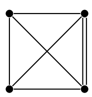
	$2(\ell-4)$	2	2
	1	2	6
	$2(\ell-4)$	2	2
	2	2	6
	$\binom{\ell-4}{2}$	2	2
	$\ell-4$	4	8
Total	$\binom{\ell+1}{4}$	$q_G=\ell^2-2\ell-3$	$\mathrm{DV}=\ell^3-2\ell^2-\ell+2$

Table 4: Computing $\mathrm{DV}(G)$ where G is a σ-hill

References

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